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STRUCTURAL ENGINEERING PROBLEMS

DEALING WITH FRAMES, WIND BRACING, RETAINING WALLS, SHEET PILING, AND WAVE PRESSURE ON BREAKWATERS

by

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PREFACE.

The present treatise represents an effort to provide solutions for problems which have not received adequate consideration in our present day literature. It should serve a useful purpose to the practicing engineer and to students in structural engineering.

The subjects dealt with are statically indeterminate beams, columns and rigid frames, especially those met with in modern buildings of steel and reinforced concrete construction. Wind bracing in tall buildings receives special consideration, while the chapters on retaining walls, sheet piling and wave pressure on breakwaters, offer a distinct contribution to engineering literature.

In the opening chapters pertaining to statically indeterminate structures, the foundation is laid for a more comprehensive understanding of structural problems, by developing methods for their diagnosis, followed by studies of principal systems, and the general laws governing them, culminating in the selection of a method of stress analysis by Mohr's work equation, which is practically universal in its application.

The solution of rigid frame problems by Castigliano's law, as generally advocated by American writers, involves the use of the differential and integral calculus. It is for this reason that the author advocates the use of Mohr's work equation by means of which, and substitution formulas (Table 26-A, here published for the first time), integrations are entirely eliminated.

Hence the method here advocated invariably leads to a solution of any problem involving redundancy, avoiding the multiplicity of methods and the confusion which has been injected into our literature through the substitution of new names for old things.

The subject of influence lines, so important in designing bridges with moving loads, and which was exhaustively treated in the author's "Kinetic Theory of Engineering Structures" (now out of print) is entirely omitted, because problems in building construction usually involve only one position of live load for maximum effects, thus rendering their use superfluous for present needs.

The volume thus comprises the following chapters:

<u>Chapter 1</u>, criteria for the diagnosis of structural problems, not exhaustively treated elsewhere.

<u>Chapter 2</u>, discusses the principal system in any structure involving redundancy, and the choice in its selection.

Chapter 3, presents the fundamental laws governing structures which every structural engineer should understand.

Chapter 4, gives the general outline leading to the solution of all structural problems involving redundancy.

Chapter 5, treats of deflection problems by area moments and Mohr's work equation.

Chapter 6, rounds out this portion of the treatise by presenting an extensive collection of problems all solved by Mohr's work equation, using substitution formulas Table 26-A

Chapter 7, on wind stresses in building frames, deals with approximate methods of designing wind systems for tall buildings, and detailing beam and column connections.

Chapter 8, on continuous concrete beams, should prove interesting and valuable in that the method advocated is simple and time saving, and leads to economical designs which have proven most satisfactory in practice.

Chapter 9, deals with earth pressure and retaining walls from a practical standpoint, giving everything that is useful and avoiding all that is unnecessary.

Chapter 10, presents problems relating to sheet piling and revetments, treated in a comprehensive manner.

Chapter 11 on wave pressure, sea walls and breakwaters, covers a subject about which much has been written without contributing much of value. The stability analysis here presented for the first time, is novel and original.

In the words of the great Maeterlink, "It is to our humblest efforts that every useful, enduring achievement of this earth is due". May this humble effort contribute something useful to the chosen profession of the Author.

STRUCTURAL ENGINEERING PROBLEMS.

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Note: Chapter 11 was printed in Trans. Am. Soc. C.E. Vol. 100 (1935) page 984.

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CHAP. I - CRITERIA FOR THE DIAGNOSIS OF STRUCTURAL PROBLEMS.

Structural Sufficiency and Redundancy.

The purpose of structures is to sustain loads and subsequently transfer them to the supporting earth or foundation.

Structures may consist of rigid frames, pin connected frames or solid web beams, or of combinations of any of these types.

When the members of a structure occupy two or more planes we speak of it as a framework in space. All of the structures which are to be considered here will consist of members in a single plane.

ART. 1. Supports external to a structure.

1. Simple contact between frictionless surfaces offers resistance in one direction only. This is exemplified in a roller bearing or hinged pendulum, and supplies one reaction condition. Fig. 1-A

2. A hinged joint or pin end connection to the earth is capable of resisting motion vertically and horizontally, yet permitting of rotation. Such a support affords two reaction conditions. Fig. 1-B.

3. A rigid support resists motion in all directions and hence involves three reaction conditions. Fig. 1-C.

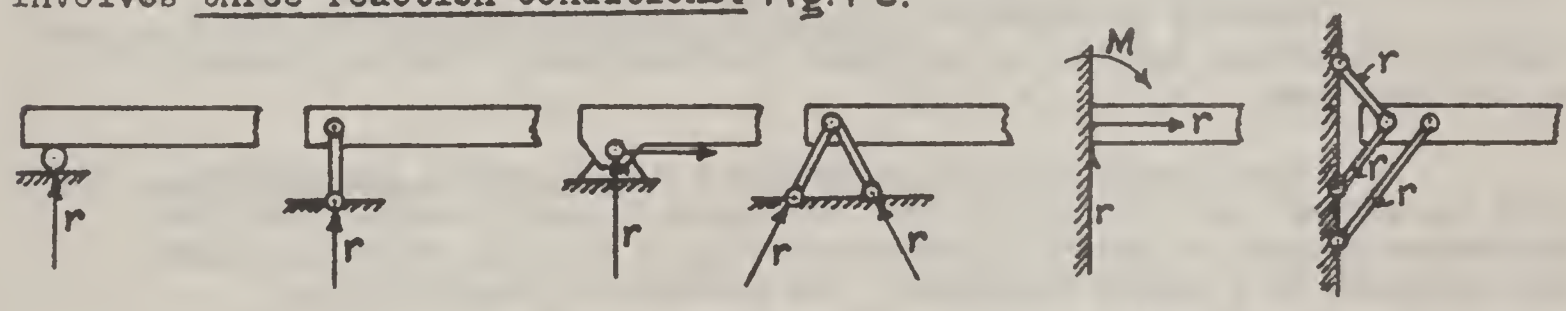


FIG. 1-A FIG. 1-B. FIG. 1-C.

These may be represented by two forces and a moment or by three forces. When the three forces are parallel the case reverts to one reaction condition, and when the three forces intersect in a commom point the case reverts to two reaction conditions.

The three types of supports, movable, hinged and fixed, thus involve respectively one, two and three reaction conditions, and each reaction condition may be represented by a short link or member capable of resisting tension or compression.

No structure, however simple, can be supported on the ground by less than three reaction conditions, this being the absolute minimum requirement for stability. When several frames are combined into a composite structure as for a three hinged arch or girder on more than two supports, or a cantilever on more than one support, then the number of necessary reaction conditions to insure stability will be :-

$$\Sigma r = 3+e-1 = e+2$$

where e is the number of elements or simple frames in the composite structure.

Hence $\Sigma r - e - 2 = n'$ represents the number of external redundant conditions for any given structure composed of e elements.

The conditions for static equilibrium applied to any structure as a whole whether simple or composite, are represented by the equations :-

$$\Sigma H = 0 \qquad \Sigma V = 0 \qquad \Sigma M = 0$$

where the summation includes all the externally applied loads including the reactions.

These same condition equations apply equally to all forces and stresses in members active about any pin point of any structure.

When the external forces are not thus balanced among themselves and with the internal stresses in its members, then the structure is unstable or statically insufficient.

A structure is therefore externally determinate when the three condition equations suffice to determine the reactions, which will also be the case when $\Sigma r = e + 2$.

A framed structure (pin connected) is internally determinate when its members are arranged to form triangles in such a manner that the successive removal of pairs of members uniting in a point, finally reduces the structure to a single triangle. The triangle is, therefore, the primary element of all determinate pin connected frames.

A structure is in every sense determinate when all reactions and stresses can be expressed purely as functions of the externally applied loads, unaffected by all temperature changes or small reaction displacements.

ART. 2. TESTS FOR REDUNDANCY OF PIN CONNECTED FRAMES.

Let p = number of pin or panel points of any given frame.
m = number of members in the frame.
n = total number of redundant conditions.
n' = number of external redundants.
n'' = " " internal "
e = number of elements or simple frames in the structure.
Σr = number of actual reaction conditions.

Then for any statically determinate structure,

$$2p = m + \Sigma r \quad \text{......................} (2A)$$

When this condition is satisfied, the frame is always statically determinate but not always stable, for it may be dynamically insufficient as will be explained later.

$$\text{When } 2p > m + \Sigma r \quad \text{......................} (2B)$$

then the structure is not in stable equilibrium and is statically insufficient.

$$\text{When } 2p < m + \Sigma r \quad \text{......................} (2C)$$

then the structure involves redundant conditions, the number of which is given by the equation

$$n = m + \Sigma r - 2p \quad \text{....................} (2D)$$

wherein a negative n would indicate static insufficiency.

In all the equations (2A) to (2D) the total number of reaction conditions for any simple or composite structure, if statically determinate, is :-

$$\Sigma r = e + 2 \quad \text{...........................} (2E)$$

The number of external redundant conditions for any given structure composed of e elements is :-

$$n' = \Sigma r - e - 2 \quad \text{.................} (2F)$$

and hence the number of internal redundant conditions must be :-

$$n'' = n - n' \quad \text{........................} (2G)$$

It is thus seen that equations (2D) and (2F) will answer every question relating to stability and redundancy of any simple or composite structure composed of pin connected triangles. The same equations also apply to solid web structures where the restrained ends constitute supports to the earth. This excludes the class of structures known as rigid frames which will now be discussed.

It should be noted that temperature stresses are produced only when there is external redundancy.

ART. 3. REDUNDANCY IN RIGID FRAMES.

In treating rigid frames it is again necessary to distinguish between external and internal redundancy. This is apparent from the fact that any rigid frame which does not close on itself may involve external redundancy due to the manner in which it is supported.

Thus two and three sided frames, also continuous and restrained beams are all externally redundant, while multiple story bents, four sided frames and many other types of rigid frames cannot be converted into a statically determinate structure simply by rearranging the supports. These involve internal redundancy in addition to the external.

The characteristic of a rigid joint as distinguished from a pin joint consists of a conventional knee brace which prevents relative rotation of one member with respect to another. Thus a pin joint will resist horizontal and vertical forces only, while a rigid joint will resist rotation as well.

As an illustration consider a four sided rigid frame Fig, 3A , on statically determinate supports.

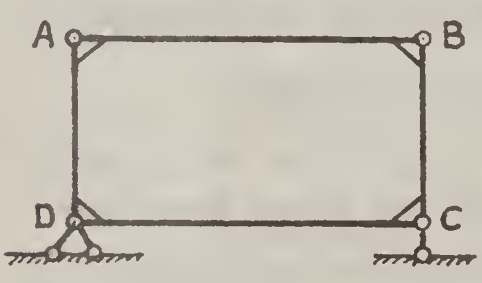


Fig.3A

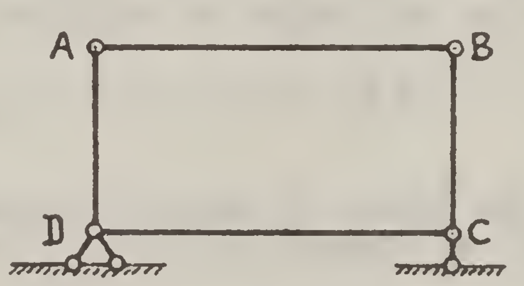


Fig.3-B.

If all four knee braces were removed, Fig. 3B, the frame would collapse, independently of how it was supported. By replacing one of the knee braces, or by inserting a diagonal member with pin end connections, stability would be again established so that we could dispense with at most three braces, and thus decide that the frame had three internal redundants.

Another illustration of a multiple three sided frame, Fig 3C, with posts hinged at the bottom, shows all the rigid joints replaced with conventional knee braces.

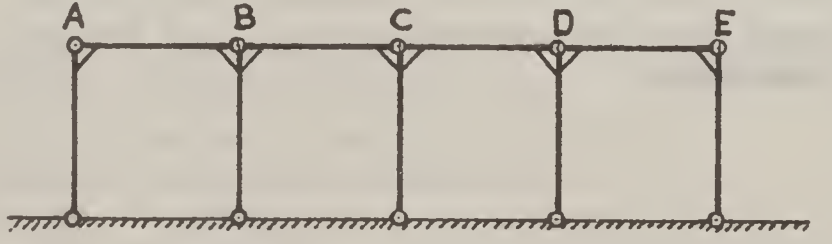


Fig.3-C.

In this frame we could dispense with seven knee braces, retaining merely one to prevent collapse of the structure. Hence we are here confronted with seven internal redundants and no external reduntants.

We will now consider a frame similar to Fig. 3C, but in which all the posts are rigidly connected to the earth by a brace at the foot of each post. Only one brace per post is necessary to produce a fixed end because the earth acts as a continuous member between posts. Fig. 3D shows this frame with all posts fixed at the bottom.

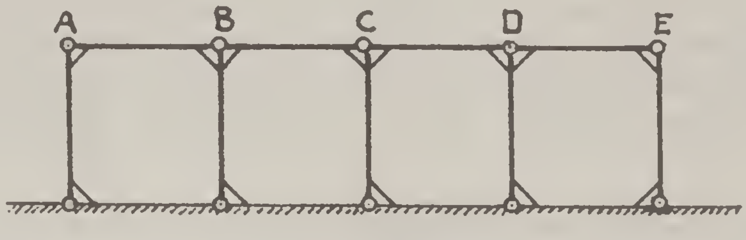


Fig. 3-D.

We could now remove three braces from each bay or a total of twelve, retaining only one brace to avoid collapse. Hence we now have a total of twelve redundants of which at least four are external if we remove all of the top braces. We would have five external and seven internal redundants if all bottom braces were removed, retaining one top brace.

Let us consider one more case of a multiple story bent as shown in Fig. 3E where all members are rigidly connected to each other and the bottom ends of the posts are fixed to the earth. We now require 34 conventional braces to exemplify all the rigid end connections involved in the frame, and 31 of these could be removed without causing collapse provided the remaining three are suitably located, as for instance those marked 1, or the three marked 2, 3, and 4, or in short retaining one brace in each story.

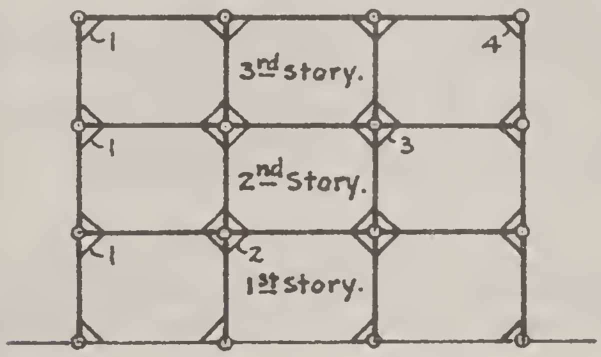


Fig. 3-E.

We may now formulate a rule for the correct diagnosis of any rigid frame, based on the number m of rigid knees and the number s of stories in the frame. The total number of redundants will be:-

$$n = m - s \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3A)$$

It is usually quite easy to count off the number n' of external redundants involved, thus leaving $n - n'$ internal redundants.

The method of successively removing the conventional knee braces from any rigid frame finally leads to a principal system as in the case of pin connected structures. This principal system forms the statically determinate portion of the rigid frame from which nothing further can be removed without causing collapse.

The introduction of the conventional knee brace offers a simple expedient in the diagnosis of rigid frame problems. One feature, however, which may be overlooked, is the case of a rigid joint combined with an actual knee brace as is often employed in bridge portals of riveted steel construction.

We then have the actual knee as an extra redundant in excess of the rigid joint with conventional knee only.

It should now be easy to contemplate the difficulties attending the analysis of many of our everyday problems in structural design, which are, however, of a purely mathematical nature, and may be solved if time permits and the endurance limit is not exceeded.

There are practical considerations and limitations which preclude the advisability of accurate analyses of most of these involved problems for the very simple reason that many uncertainties exist which greatly outweigh or overshadow relatively large approximations which may generally be permissible in expediting our work.

A few of the commonly neglected elements which affect our structures, sometimes on the side of safety and at other times adversely or even dangerously, may deserve mention here.

For instance, there is no such thing as a frictionless pin connection, yet this is commonly assumed in our computations of pin connected frames. On the other hand, there is no such thing as complete restraint which is a physical impossibility, yet we regard so-called riveted connections and concrete frames as rigid. The truth of the matter is evidently some intermediate status of fixity in both cases, and may differ widely from our assumptions in calculating stresses.

In reinforced concrete structures we encounter a variety of uncertainties which defeat all efforts toward accuracy in design. Thus in buildings the unavoidable restraint between floor-slabs and beams make it impossible to estimate the effective cross section of T beams. Also, the restraint between beams, girders and columns introduces uncertainties which must render our calculations grossly approximate.

Even in steel frames a considerable restraint is always developed by the standard end connections to other beams and columns, and yet we base our calculations on simple beam assumptions.

Under such circumstances we are amply justified in adopting approximate methods of design so long as our errors are on the side of safety instead of bordering on recklessness.

With these facts in mind we can usually simplify the solution of complex problems by neglecting those factors which are of minor importance, but add greatly to the labors of the designer. However, to exercise this criterion with prudence, we must acquire a maturity of judgement which canbe attained only in the workshop of experience.

ART. 4. PROBLEMS ILLUSTRATING STRUCTURAL TYPES.

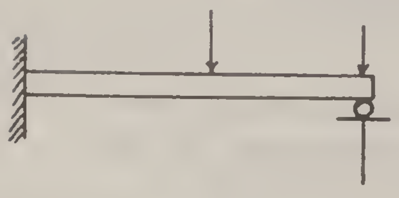


Fig. 1.

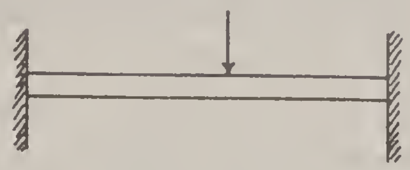


Fig. 2.

Fig. 3.

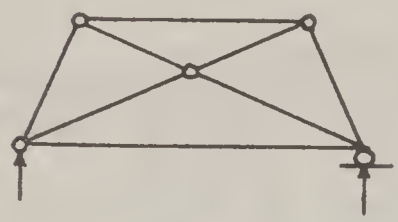


Fig. 4.

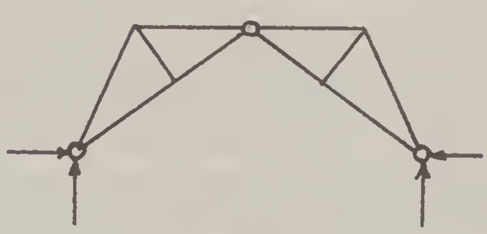


Fig. 5

Fig. 6.

Fig. 7.

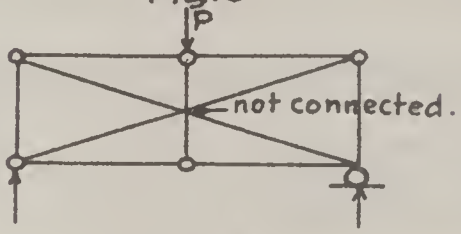


Fig. 8.

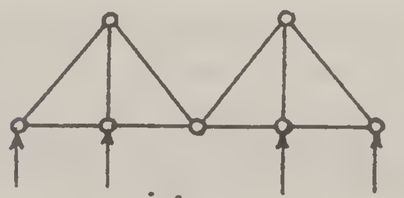


Fig. 9.

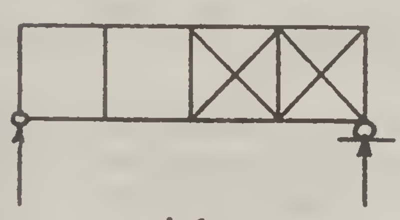


Fig. 10.

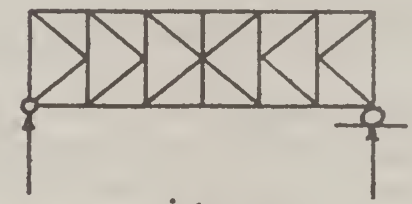


Fig. 11.

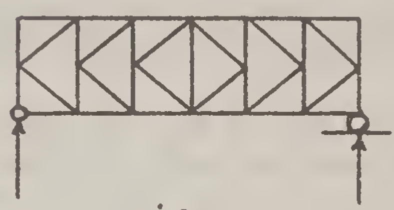


Fig. 12.

Fig. 13.

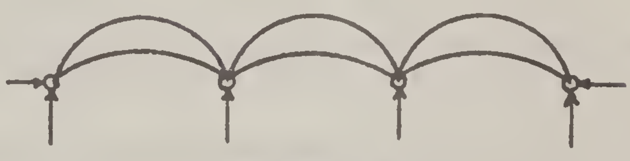


Fig. 14.

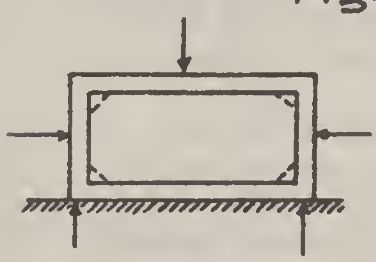


Fig. 15.

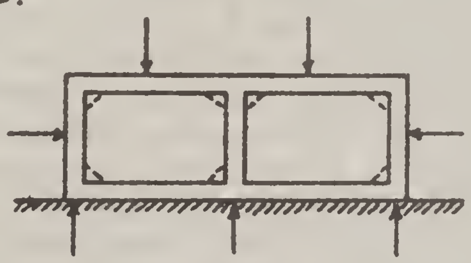


Fig. 16.

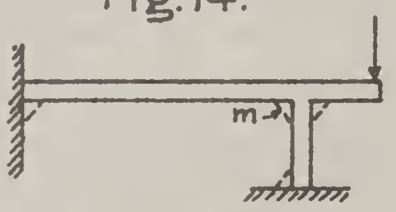


Fig. 17.

$n = \Sigma r + m - 2p$

Fig	Σr	m	p	e	Remarks
1	4	1	2	1	1 ext. redundant
2	6	1	2	1	3 " "
3	6	1	2	1	3 " "
4	3	8	5	1	1 int. "
5	4	10	7	2	Stat. determ.
6	4	23	13	1	1 ext. redund.

$n' = \Sigma r - e - 2$

Fig	Σr	m	p	e	Remarks.
7	6	19	11	1	3 ext. redundants
8	3	9	6	1	infinite stress.
9	8	10	7	2	4 ext. redundants
10	3	17	10	1	Unstable
11	3	36	19	1	1 int. red.
12	3	37	20	1	Stat. determ.

$n = m - 1$ for 1-story frame.

Fig	Σr	m	p	e	Remarks.
13	5	45	24	1	2 ext. redundants
14	8	3	4	3	3 " "
15	3	4	–	–	3 int. "
16	4	7	–	–	6 int. "
17	6	4	–	–	3 ext. "

CHAP. 2 - The Principal System of a Structure Involving Redundancy.

Art. 5. The criteria for recognizing the statically determinate portion or principal system of any structure involving redundancy, were discussed in the previous chapter. We will now investigate the nature and form which a principal system may assume for any redundant conditions as may be found to exist in any given structure.

A principal system may be defined as any statically determinate portion of a structure from which all external and internal redundant conditions have been removed. The conception of a principal system affords a means for the comprehensive and universally applicable method of analysis of all problems involving redundancy. It was first introduced by Prof. Otto Mohr in 1874, and constitutes the basis for his well known work equations.

Having ascertained the number of redundants involved in a given example, by applying our criteria, we then proceed to remove them so as to arrive at some suitable principal system. We then reapply these several assumed redundants along with the externally applied loading, to ascertain the combined effect of all on whatever principal system was selected.

The principal system being statically determinate, permits of finding all functions pertaining thereto by the ordinary methods of statics, for the severally assumed redundants and the actual externally applied loading.

Hence, if we know how to find reactions, moments, shears, stresses and deflections for any statically determinate structure, then we possess the tools for solving redundancy problems by applying Mohr's work equation to the principal system. This will be dealt with in Chapter four.

Our next effort will be directed to the choice of a principal system for any given problem, for much depends on making a judicious choice in order that our subsequent labors may be minimised.

In all our investigations of statically indeterminate structures the redundant conditions, whether reaction forces, moments, shears or stresses in members, will always be designated by X_a, X_b, X_c, etc, implying that they represent those unknowns which must first be evaluated from the elastic properties of the materials of construction, and the geometric relations presented by the structural dimensions. The actual evaluation of these redundnats X is accomplished by means of elasticity equations according to Mohr, of which there will always be as many as the number of redundants. That is, one work equation can be written for each redundant thus furnishing as many equations as there are unknowns X. With the X's evaluated, the remainder of the analysis reverts to the ordinary methods of statics as applied to the principal system, wherein the X's are treated like known external forces.

The particular reactions or members best suited to represent the redundant conditions designated by X_a, X_b, X_c, etc., are those which reduce the given structure to the simplest possible principal system.

Any member or reaction of an indeterminate structure may be removed to produce the principal system so long as the latter still remains a stable determinate structure.

However, the rule should be to select a principal system of the simplest possible form, always avoiding composite structures such as a three-hinged arch or a cantilever system, whenever a simple beam, cantilever, or truss could as well be chosen.

Solid web structures should always be transformed into externally determinate beams by assigning the redundant conditions to the supports.

Framed structures, if externally indeterminate, should always be so transformed as to remove the external conditions. The only exception to this rule might be a continuous girder wherein a top chord member over each intermediate pier might be treated as a redundant member.

When the redundant conditions are internal, then the only way of deriving the principal system is to remove such redundant members and replace each one by two equal and opposite forces X.

Art. 6. Problems illustrating Principal Systems.

A number of problems will be presented here with a view of showing the various possibilities in the choice of a principal system, and the manner in which any redundants may be disposed of by means of conventional loadings, reapplied to the principal system.

Problem 6 A. Simple beam fixed at one end and supported on a roller bearing at the other end.

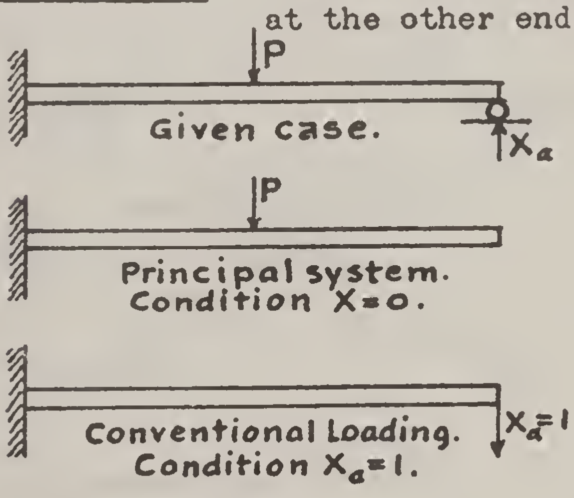


Fig.(a).

$\Sigma r = 4$, $m = 1$, $p = 2$, $e = 1$

$$n = \Sigma r + m - 2p = 1$$

$$n' = \Sigma r - e - 2 = 1$$

One external redundant.

a. Principal system a cantilever, obtained by removing the roller bearing and calling the end reaction the redundant X_a. Condition X = 0. The conventional loading $X_a = 1$, applied to the principal system to the exclusion of all other loads. Condition $X_a = 1$.
When X_a is found for the actual case of external loading, then every other feature of the problem is solvable by the methods known to statics.

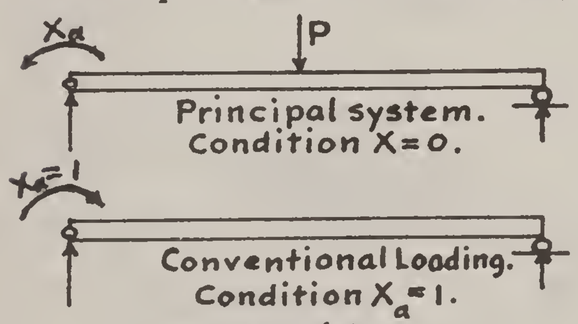
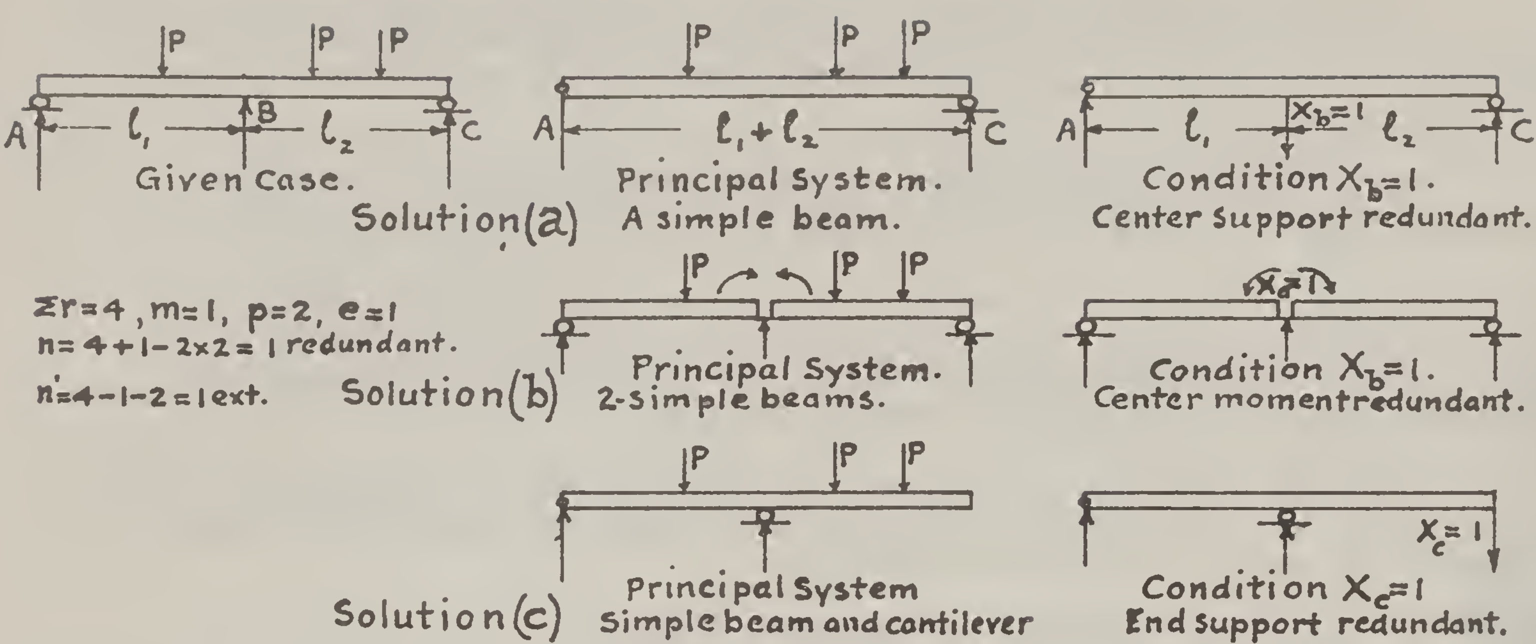


Fig.(b).

b. Principal system a simple beam obtained by converting the fixed end support into a pin end support with an external redundant moment. Condition X = 0. The conventional loading $X_a = 1$ is applied to the principal system and pictures the condition $X_a = 1$.
When X_a is found for the actual case of external loading, the problem is again solvable by statics and will lead to answers which are numerically identical with the cantilever solution. There is not much choice between these two solutions as they are equally simple.

Problem 6 B. Beam on three supports.

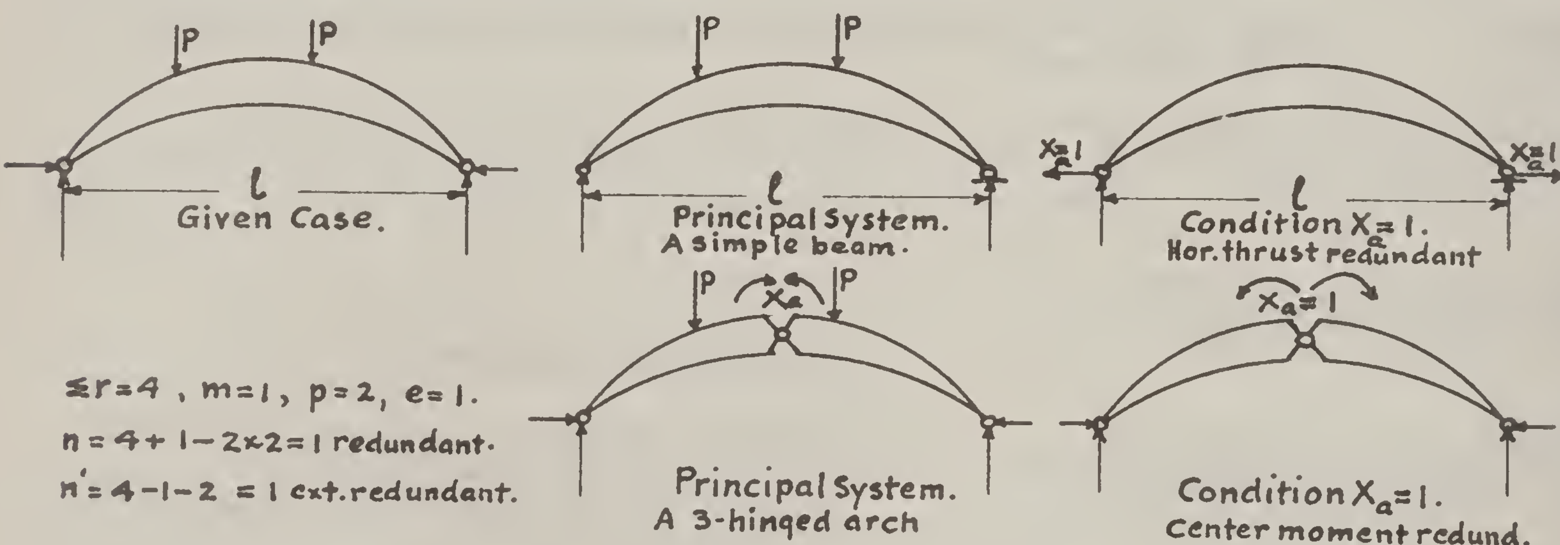


$\Sigma r = 4$, $m = 1$, $p = 2$, $e = 1$

$n = 4 + 1 - 2 \times 2 = 1$ redundant.

$n' = 4 - 1 - 2 = 1$ ext.

Problem 6 C. Two hinged arch.

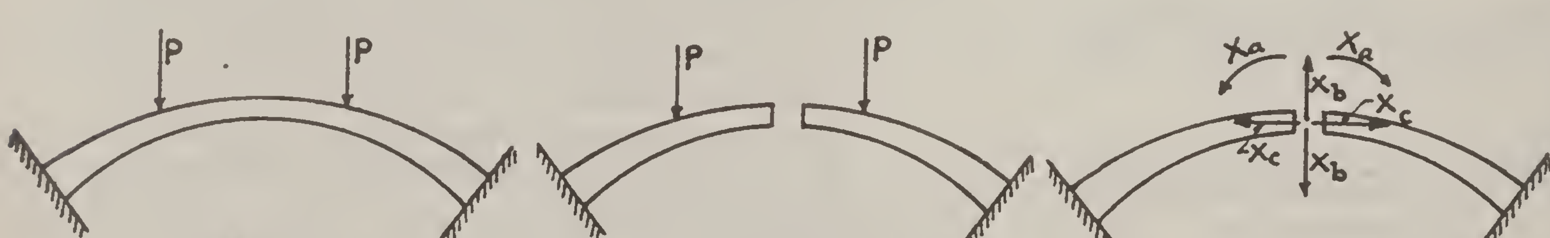


$\Sigma r = 4$, $m = 1$, $p = 2$, $e = 1$.

$n = 4 + 1 - 2 \times 2 = 1$ redundant.

$n' = 4 - 1 - 2 = 1$ ext. redundant.

Problem 6 D. Fixed solid web arch rib.



$\Sigma r = 6$, $m = 1$, $p = 2$, $e = 1$.

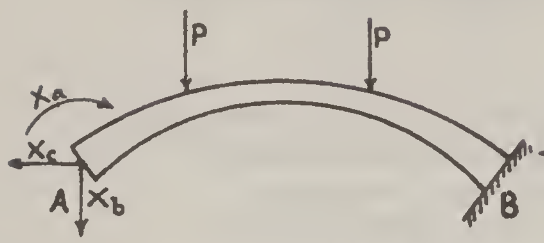
$n = 6 + 1 - 2 \times 2 = 3$ redundants.

$n' = 6 - 1 - 2 = 3$ external.

3 redundants; Vert. Shear X_b
hor. thrust X_c
Moment X_a
Applied to each Cantilever in opposite sense for each arm.

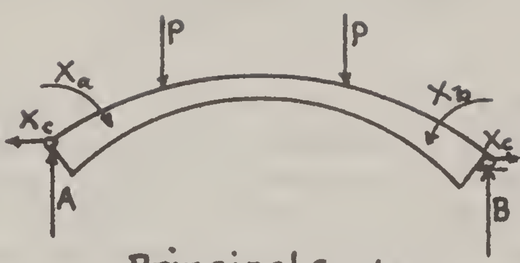
Solution (a).

Problem 6 D. continued.



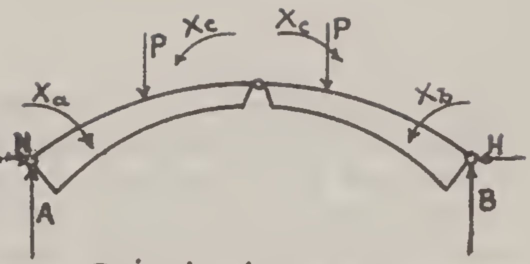
Principal system
Simple Cantilever.
3 redundants; Moment X_a
Hor. thrust X_c
Vert. React. X_b

Solution (b)



Principal system
Simple beam.
3 redundants; Moment X_a
Moment X_b
Hor. thrust X_c

Solution (C)



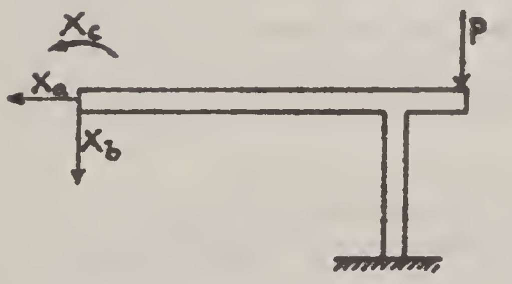
Principal system
3-Hinged arch.
3 redundants; Moment X_a.
Moment X_b.
Moment X_c.

Solution (d).

Case c gives preferable solution.

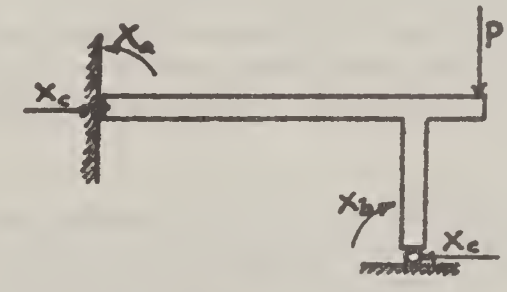
Problem 6 E. Fixed Beam and Column with Cantilever.

$\Sigma r = 6$, $m = 1$, $p = 2$, $e = 1$.
$n = 6 + 1 - 2 \times 2 = 3$ redundants.
$n' = 6 - 1 - 2 = 3$ ext. redundants.



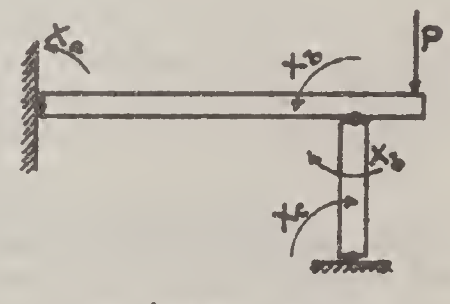
Principal system
Simple Cantilever.
3 redundants; Hor. thrust X_a
Vert. React. X_b
Moment X_c

Solution (a)



Principal system
Simple beam.
3 redundants; Moment X_a
Moment X_b
Hor. thrust X_c

Solution (b)



Principal system
3-Hinged arch.
3 redundants; Moment X_a.
Moment X_b.
Moment X_c.

Solution (C).

Case c gives preferable solution.

Chapter 3. Fundamental Laws

Governing Frames and Solid Web Structures, both statically determinate and indeterminate.

Art. 7. Introductory. The laws of statics suffice for the analysis of all statically determinate structures in so far as stresses are concerned, and the elastic properties of the material are not involved. This method of treatment tacitly assumes that the structure is at rest. However, the moment when externally applied loads begin to act on a structure it ceases to be at rest, and must undergo elastic deformation so that we really have no license to regard an elastic structure as statically at rest only when it is neither stressed nor distorted.

In abroader sense we must regard all structures as mechanical contrivances subject to some elastic motion, instead of inelastic bodies at rest. It is true that this motion prevails only while a change in the elastic balance is taking place, as when loads are added to, or removed from, the structure, but this may apply to any mechanism of interrupted activity.

Therefore, while we speak of a structure as in static equilibrium, we may also speak of it as in a state of dynmanic equilibrium, a state which the structure assumes in the instant that the super-imposed loads do not produce any further deformations. The same would be true when all loads are entirely removed, in which case the dynamic equilibrium returns to the special case of static equilibrium. This is merely a broader view point involving the principles of work, embracing at once all of the conditions as they really exist in a structure sustaining loads.

For the same structure, the magnitude of the deformation is a direct function of the applied loads, the unit stresses which they produce, and the elastic properties of the materials employed.

The stiffness or rigidity of a given structure, is however, independent of the magnitudes of the unit stresses, but depends entirely on the ability of the structure to resist stress, and this in turn is a function of the elasticity of the material and the geometric shape of the structure.

With these concepts in mind, the general laws pertaining to the comprehensive analysis of structures and their behavior under loads will now be considered in brief review even though not all of them will necessarily be applied to the solution of problems.

The work of deformation constitutes the basis for the solution of all those problems which involve the elastic properties of the material and which are not susceptible to analysis by the methods of pure statics.

Art. 8 Law of elastic deformations

Let $S =$ total stress in any member of a frame from any cause designating tension by +
$l =$ length of this member when $S = 0$.
$\Delta l =$ change in length l due to stress S, + for elongation.
$F =$ cross-section of this member, uniform over length l.
$t =$ a uniform change in temperature in degrees + for rise.
$\varepsilon =$ coefficient of linear expansion per degree temperature.
$E =$ modulus of elasticity = unit stress divided by unit strain.
$f = S/F$ = unit stress in the member
$l/EF = \rho$ = the extensibility.
$\Delta l/l =$ unit strain.

Then according to Hook's law,

$$E = \frac{f}{\frac{\Delta l}{l}} = \frac{S/F}{\Delta l/l},$$

from which $\Delta l = \frac{Sl}{EF} + \varepsilon t l$ ....(8A)

where $\varepsilon t l$ represents the added change in length due to temperature.

This equation represents the elastic deformation for any member of any frame and is a fundamental elasticity condition.

Art 9 Law of the summation of similar partial effects.

Every set of simultaneous causes or conditions, such as a load P, temperature rise t, and reaction displacement Δr, acting on a given structure, produces a partial stress S' which contributes in producing an ultimate total value S due to all partial values or effects resulting from the respective sets of independent causes or conditions.

This is expressed algebraically by

$$S = S' + S'' + S''' + \text{ etc} \quad \text{.................................(9A)}$$

and signifies that each effect, such as a stress, whether due to loads, temperature, or abutment displacements, may be ascertained or investigated by itself, and the sum total effect S will then be the sum of the several similar partial effects.

This law is fundamental to the analysis of all structures involving redundancy.

Art 10 Law of proportionality between cause and effect.

Equation (9A) being true for any set of effects, would remain true for any multiple of these effects. Thus if a set of loads P produces stresses S in the members of a given frame, then another set of parallel loads $2P$ acting at the same points, would produce stresses $2S$ in the same members. Or, if a single load unity, acting on a point m of any frame produces reactions R_1, stresses S_1, and deflections δ_1, then a load P_m parallel to the unit load and acting on the same point m will produce reactions $R = P_m R_1$, stresses $S = P_m S_1$, and deflections $\delta = P_m \delta_1$.

Art 11 Law of summation with redundants.

This law expresses the summation of all partial effects due respectively to the applied loads, the several redundants, and to temperature changes, when acting simultaneously on the principal system of any pin connected frame or solid web structure. The sum total effect sought may be the stress in any member, the reaction at any support, or the bending moment about any point of the structure, and the partial effects contributing to any one total must necessarily be similar in kind or character.

The structure under consideration must be so constituted that the removal of all the redundant conditions will produce a statically determinate structure with the necessary supports. The statically determinate structure thus remaining will always be called the principal system. The redundants, when applied to the principal system, will always be designated as unknowns X_a, X_b, X_c, etc, and when singly applied to the principal system the several cases of loading will be spoken of as conventional loadings.

When the redundants X (known or unknown) are again applied to the principal system, together with the other external loads P, then the resulting effect on every portion of the principal system must be identical with the effect which would have been produced by the original loading of the given indeterminate structure. This will also be true of deformations.

The following definitions of terms will be strictly adhered to in all succeeding discussions.

Let $S =$ the stress in any member of a principal system.
$S_o =$ the stress in this member due to loads P when the several redundants X, and t are all zero, to be known as Condition X = 0
$S_a =$ the stress in this member of the principal system when no load other than $X_a = 1$ is active. Condition $X_a = 1$.
$S_b =$ the same when no load other than $X_b = 1$ is active on the principal system. Condition $X_b = 1$.
$S_c =$ the same for Condition $X_c = 1$.
$S_t =$ the same for a uniform change in temperature.

R_o, R_a, R_b, R_c etc, are defined like the S's with like subscripts, but represent reactions.

M_o, M_a, M_b, M_c etc, are moments defined like the S's with like subscripts.

X_a, X_b, X_c, etc, are the redundants, and may be stresses in members, reactions, moments, etc, as may be assigned in reducing the given case to a principal system.

$\delta_a, \delta_b, \delta_c$, etc, are the displacements in the points of application of the redundants, for effects due to loads P and redundants X.

The stress S in any member of a frame involving redundancy, is a linear function of the loads P, X_a, X_b, X_c, etc, all treated as external forces applied to the principal system. This follows because all conditions of equilibrium are represented by linear equations whether dealing with stresses, moments, or reactions.

Now, according to the law of the summation of similar partial effects expressed by equation (9A) the general equation for stress in any member of a frame involving redundancy, would have the form

$$S = S_o - S_{ax} - S_{bx} - S_{cx} - \text{etc.} + S_t ,$$

wherein $S_{ax} =$ the stress in this member due to X_a acting alone on the principal system, and $S_{bx} =$ the stress in this member due to X_b acting alone on the principal system, etc.

The negative signs applied to the stresses produced by the redundants indicate that these stress contributions are always of opposite sign to the stress contributed by external loads P . This follows because the increment of work performed by the redundants X is always negative with respect to the work performed by the stress S_o, because the redundants are classed with the external forces but actually assist in carrying them.

Also, from the law of proportionality if follows that $S_{ax} = S_a X_a$, $S_{bx} = S_b X_b$, etc. Hence for the stress in any member of the principal system

$$S = S_o - S_a X_a - S_b X_b - S_c X_c \text{ etc.} + S_t$$

also for any reaction of the principal system ...(11A)

$$R = R_o - R_a X_a - R_b X_b - R_c X_c \text{ etc.} + R_t$$

and for any moment on the principal system

$$M = M_o - M_a X_a - M_b X_b - M_c X_c \text{ etc.} + M_t$$

In these equations the quantities S_o, R_o, and M_o are all linear functions of the externally applied loads P, while all the stresses, reactions, and moments bearing subscripts a, b, c, etc, are constants due to conventional loadings and are absolutely independent of all external loads P and X applied to the principal system.

These equations are fundamental in the analysis of all structures involving redundancy.

Art. 12. General Work Equation. Clapeyron's Law, 1833.

The work of deformation of any structure sustaining loads, constitutes the basis for the solution of all those problems which involve the elastic properties of the materials and which are not susceptible to analysis by the methods of pure statics.

A loaded frame is a machine in equilibrium and within the limits of proportional elasticity its deformation varies directly with the magnitude of the superimposed loads.

The question of deformation does not enter into statics and hence a general and comprehensive treatment of the elastic frame must necessarily involve the principles of mechanics.

A given frame with definite loading and supports constitutes a system in external equilibrium. The frame undergoes certain deformations which steadily increase in direct proportion with the internal stresses created in the members by the external forces as they are gradually applied. The final deformation will occur in the instant when the external forces are exactly balanced by the internal stresses, when a condition of static equilibrium is attained.

During the process of deformation, the applied loads travel through distances which are the displacements of their points of application, producing the positive work of deformation of the external forces.

The internal stresses in the members must accomodate themselves to the deformed condition of the frame and in resisting this action must produce negative work of deformation.

In the instant when static equilibrium is established between the loads and the stresses, the positive and negative work of deformation, produced in the same time interval, must exactly balance, according to the doctrine of the conservation of energy.

A positive amount of work is always produced when a force and its displacement act in the same direction.

The product of $\frac{1}{2}P\delta$ represents the actual work produced by a force gradually applied and increasing from its initial zero value to a certain end value P, thus exerting only its average intensity during the entire time of traversing the path to perform this work. Hence, for all the external forces P acting on any frame, the total <u>positive external work of deformation</u> would be :-

$$A_e = \tfrac{1}{2}\Sigma P\delta.$$

Similarly $\frac{1}{2}S\Delta\ell$ represents the actual internal work resisted by any member subjected to a gradually increasing stress of end value S, with an average intensity S/2 during the entire process while producing a change in its length of $\Delta\ell$. Hence the stresses in all members in any frame will represent a total <u>negative work of deformation</u> of :-

$$A_i = \tfrac{1}{2}\Sigma S\Delta\ell.$$

As the applied work must equal the work overcome, therefore, for any frame,

$$\tfrac{1}{2}\Sigma P\delta = \tfrac{1}{2}\Sigma S\Delta\ell = \tfrac{1}{2}\Sigma S^2\frac{\ell}{EF}, \qquad \text{.....................(12A)}$$

which is Clapeyron's law and may be stated as follows :-

For any frame of constant temperature, and acted on by loads which are gradually applied, the actual work produced during deformation is independent of the manner in which these loads are created and is always half as great as the work otherwise produced by forces retaining their full end values during the entire act of deformation.

By regarding any elastic body as composed of an infinite multiplicity of members, it is readily seen that Clapeyron's law applies equally to frames and solid web elastic structures when properly supported.

Cases involving dynamic impact would imply a certain amount of kinetic energy in excess of the negative work of deformation; that is, the applied forces have some initial value, greater than zero, at a time when the stresses are still zero. Hence equation (12A) does not apply to effects of falling bodies.

Art. 13. Law of Virtual Work. 1788. Lagrange.

This law was first enunciated by Galilei and Stevin and later by Bernoulli, but Lagrange first reduced it to an algebraic expression which permits of a more varied application than Clapeyron's law. The following derivation may be of interest.

All forces or stresses, acting on a pin point of any structure in equilibrium, will have components parallel to any axis, and the sum of such a set of parallel components must be zero. Calling α the angle which any force or stress Q makes with the axis chosen, then for all such meeting in one point,

$$\Sigma Q \cos\alpha = 0$$

Now if a displacement Δ, parallel to the axis, be arbitrarily assigned to this pin point, and assuming equilibrium to continue, then the product of Δ with the sum of the components must still be zero. This is equivalent to multiplying both sides of the above equation by Δ to obtain :-

$$\Sigma (Q \cos\alpha)\Delta = 0$$

But the displacements $\Delta \cos\alpha = \delta$ are the projections of the displacement Δ on the directions of the several forces, hence :-

$$\Sigma Q\delta = 0 \qquad \text{(13A)}$$

wherein the forces Q are in every sense independent of the displacements δ, which latter may be any possible displacements resulting from some state of equilibrium due to another set of forces P in the same complex of members.

Hence for any point, frame, or body acted on by loads Q in an established state of equilibrium, the sum total work performed by these forces, in moving over any small arbitrary but possible displacements δ, must always be equal to zero.

Art. 14. Mohr's Work Equations. 1874.

Proceeding from the law of virtual work as applied to two cases of loading, once for arbitrary loads Q producing stresses S_a in equilibrium, and once for displacements δ and changes in the lengths of members Δl due to loads P in equilibrium, we can write for any frame :-

$\Sigma Q\delta = 0$ and $\Sigma S_a \Delta l = 0$, and hence by subtraction:-

$$\Sigma Q\delta - \Sigma S_a \Delta l = 0 \qquad \text{(14A)}$$

which represents a condition of **elastic equilibrium** as distinguished from static equilibrium.

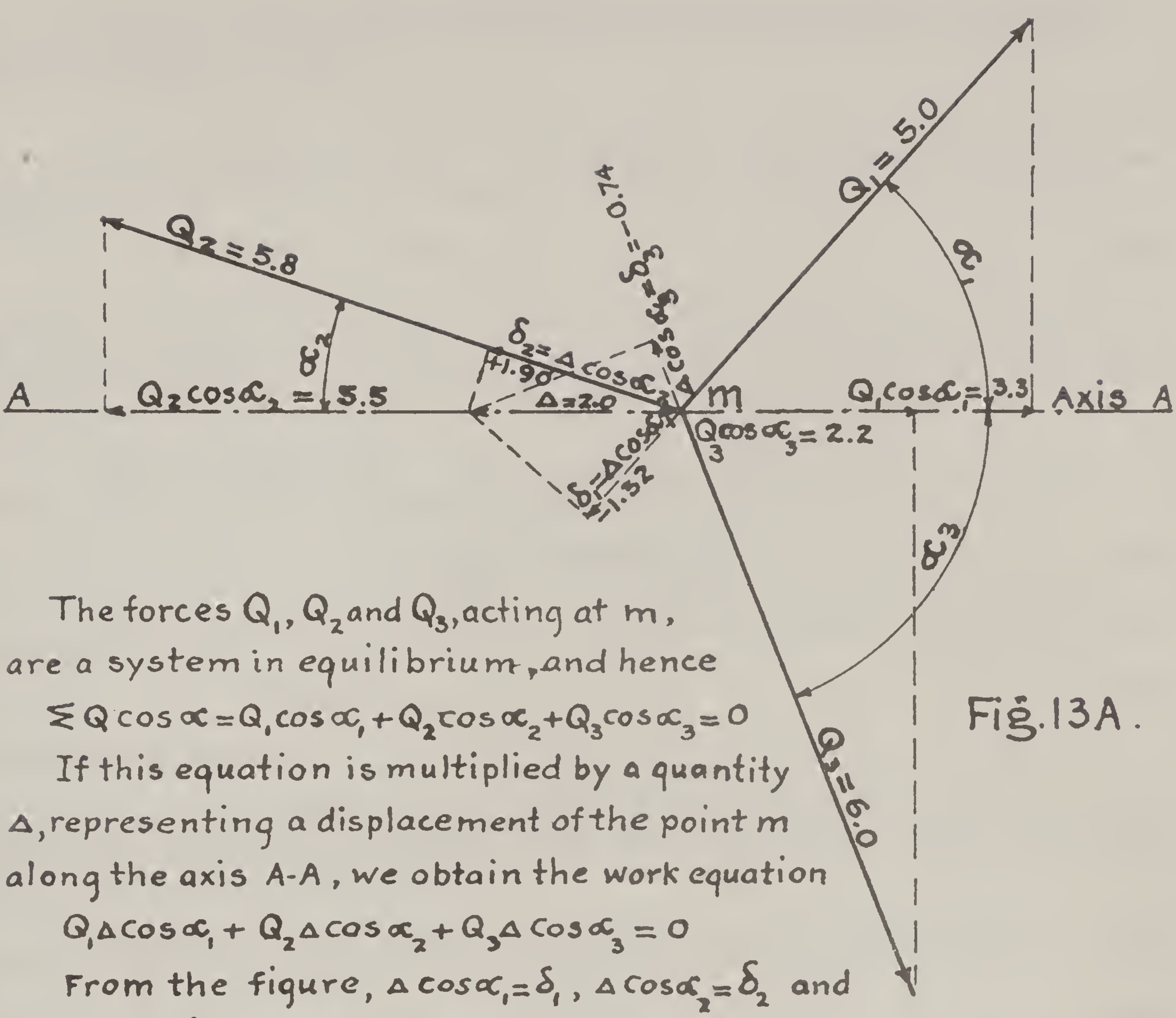


Fig. 13A.

The forces Q_1, Q_2 and Q_3, acting at m, are a system in equilibrium, and hence

$$\Sigma Q\cos\alpha = Q_1\cos\alpha_1 + Q_2\cos\alpha_2 + Q_3\cos\alpha_3 = 0$$

If this equation is multiplied by a quantity Δ, representing a displacement of the point m along the axis A-A, we obtain the work equation

$$Q_1\Delta\cos\alpha_1 + Q_2\Delta\cos\alpha_2 + Q_3\Delta\cos\alpha_3 = 0$$

From the figure, $\Delta\cos\alpha_1 = \delta_1$, $\Delta\cos\alpha_2 = \delta_2$ and $\Delta\cos\alpha_3 = \delta_3$, therefore we may write

$$\Sigma Q\delta = Q_1\delta_1 + Q_2\delta_2 + Q_3\delta_3 = 0.$$

Add at the end of Art. 13, p. 16

This law expresses the equality between the external and internal work of deformation for real displacements and arbitrary cases of loading, provided equilibrium exists.

By allowing all the arbitrary loads Q in equation (14A) to vanish, and substituting therefor a single load unity at a point m, producing stresses S_a (and reactions R_a) in the members of a frame, then the new work equation becomes :-

$$1.\delta_m + \Sigma R_a \Delta r = \Sigma S_a \Delta l \text{ .... (14B)}$$

wherein δ_m is the displacement of the point of application of the unit load at m in the direction of this load, and Δl successively for each member, is the change in length of such member, due to loads P producing a set of stresses S in the members, and abutment displacements Δr at the supports.

Since $\Delta l = \frac{Sl}{EF} + \varepsilon t l$ when temperature effects are also included, then equation (14B) becomes in its most comprehensive form

$$1.\delta_m = \Sigma S_a\left(\frac{Sl}{EF} + \varepsilon t l\right) - \Sigma R_a \Delta r, \text{ .... (14C)}$$

Omitting temperature effects and abutment displacements, we have the usual form

$$1.\delta_m = \Sigma S_a \frac{Sl}{EF}, \text{ .... (14D)}$$

for Mohr's work equation.

The unit loading producing stresses S_a and reactions R_a will always be called the <u>conventional loading</u> and may be a single force unity, or a moment equal to unity, applied at a point m.

For solid web structures equation (14D) must be transformed to include the effects due to direct stress and bending, and then becomes :-

$$1.\delta_m = \frac{1}{EF}\int_0^l N N_a dx + \frac{1}{EI}\int_0^l M M_a dx, \text{ .... (14E)}$$

wherein N = the direct axial stress due to loads P
M = the bending moment at any point due to loads P
N_a = the direct axial stress due to a conventional unit loading at m
M_a = the bending moment at any point due to the conventional unit loading at m.
δ_m = displacement of point m in the direction of the unit conventional load, and due to actual loads P.

<u>For a single load P_m</u>, the deflection of the point m in the direction of P_m becomes by equations (14D) (14E)

$$P_m \delta_m = \Sigma \frac{S^2 l}{EF} = \frac{1}{EI}\int_0^l M^2 dx$$

or,

$$\delta_m = \frac{1}{P_m} \Sigma \frac{S^2 l}{EF} = \frac{1}{P_m EI}\int_0^l M^2 dx \text{ .... (14F)}$$

Art 15. Maxwell's Law. 1864.

The theorem generally known as the <u>law of reciprocal displacements</u> establishes the mutual relation between the elastic displacements of any two points of any structure, whenever these displacements result from simultaneous conditions of loading; provided that the members remain unchanged and the supports are immovable.

To simplify the proof of this law let us deal with two points m and n of any framed structure, and let δ_{mn} = displacement of the point m when only one load $P_n = 1$ is acting at the point n, producing stresses S_n. Also let δ_{nm} = displacement of the point n when only one load $P_m = 1$ is acting at the point m, producing stresses S_m.

Then applying Mohr's work equation (14D) to the two cases of loading, we have :—

Deflection at $m = \delta_{mn} = \Sigma S_m \frac{S_n \ell}{EF}$ for $P_n = 1$ at point n.

Deflection at $n = \delta_{nm} = \Sigma S_n \frac{S_m \ell}{EF}$ for $P_m = 1$ at point m.

Since $\Sigma S_m \frac{S_n \ell}{EF} = \Sigma S_n \frac{S_m \ell}{EF}$, therefore :-

$$\delta_{mn} = \delta_{nm} \qquad \text{.......................(15A)}$$

This is Maxwell's law and may be stated thus :-
The displacement δ_{mn} of a point m in a direction mm', produced by a unit load acting at n in another direction nn', is equal to the displacement δ_{nm} of a point n in the direction nn', produced by a unit load acting at m in the direction mm'.

Art. 16. Menabrea's Law. 1858. or theorem of least work.

For any statically indeterminate frame, the redundant conditions reduce the actual work of deformation to a minimum.

Now the actual work of deformation for any frame including redundant members is given by Clapeyron's law) Equation (12A) as :-

$$A = \frac{1}{2} \Sigma S^2 \frac{\ell}{EF} \qquad \text{.......................(16A)}$$

and this must become a minimum when the stresses S have the values given by Equation(11A) as :-

$$S = S_o - S_a X_a - S_b X_b - S_c X_c, \text{etc.,} \qquad \text{.......................(16B)}$$

Differentiating Equation (16A) gives :-

$$\partial A = \Sigma \frac{\ell}{EF} S \partial S \qquad \text{.......................(16C)}$$

Also the partial differentiation of S with respect to each X in Equation (16B) gives :-

$$\frac{\partial S}{\partial X_a} = -S_a ; \quad \frac{\partial S}{\partial X_b} = -S_b ; \quad \frac{\partial S}{\partial X_c} = -S_c ; \text{ etc.,} \qquad \text{.......................(16D)}$$

Dividing both sides of Equation (16C) by ∂X_a and substituting $\frac{\partial S}{\partial X_a} = -S_a$, gives :-

$$\frac{\partial A}{\partial X_a} = \Sigma \frac{\ell}{EF} S \frac{\partial S}{\partial X_a} = -\Sigma \frac{\ell}{EF} S_a S, \qquad \text{.......................(16E)}$$

which must be equal to zero if there are no reaction displacements, since by Mohr's work equation the displacement $\delta_a = \Sigma S_a \frac{S\ell}{EF}$, for the point a, which is the point of application of X_a.

Similarly then,

$$\frac{\partial A}{\partial X_a} = 0 ; \quad \frac{\partial A}{\partial X_b} = 0 ; \quad \frac{\partial A}{\partial X_c} = 0 ; \text{ etc.}$$

which proves that the redundants reduce the actual work of deformation to a minimum.

Art 17. Castigliano's law (1879), or derivative of the work equation. It deals with the displacement of the point of application of a force.

The actual work of deformation due to a load P_m acting at m on any frame, is by Clapeyron's law

$$A_e = \tfrac{1}{2} P_m \delta_m \quad \text{.........(17A)}$$

where δ_m is the displacement within the range of proportionality, of the point m in the direction of P_m

Also, the actual internal work of deformation for the entire frame is by the same law

$$A_i = \tfrac{1}{2} \Sigma \frac{S^2 \ell}{EF} \quad \text{.........(17B)}$$

The partial differential derivative of A with respect to any external load P_m is from (17B)

$$\frac{\partial A}{\partial P_m} = \frac{1}{2} \Sigma \frac{\partial}{\partial P_m}\left(\frac{S^2 \ell}{EF}\right) = \Sigma S \frac{\partial S}{\partial P_m}\left(\frac{\ell}{EF}\right), \quad \text{.........(17C)}$$

wherein $\partial S / \partial P_m$ is the derivative of the stress S in any member of a determinate or indeterminate frame, and may be evaluated from the general equation (11A). Noting that $S_o = S_m P_m$ for a single load P_m, then equation (11A) gives the total stress S for P_m and redundants X as :

$$S = S_m P_m - S_a X_a - S_b X_b - S_c X_c, \, etc. \quad \text{.........(17D)}$$

The stress S_m in any member S for $P_m = 1$, is independent of the X's, and the X's are independent of the load P_m, hence S may be partially differentiated with respect to P_m or any X, and equation (17D) when so treated gives

$$\frac{\partial S}{\partial P_m} = S_m \quad \text{and} \quad \frac{\partial S}{\partial X_a} = -S_a, \, etc., \quad \text{.........(17E)}$$

Substituting S_m for $\partial S / \partial P_m$ in equation (17C) then

$$\frac{\partial A}{\partial P_m} = \Sigma S_m S\left(\frac{\ell}{EF}\right) \quad \text{.........(17F)}$$

But Mohr's law, equation (14D) gives a value for deflection δ_m as follows:-

$$1.\delta_m = \Sigma S_m S \frac{\ell}{EF} \quad \text{.........(17G)}$$

Therefore,

$$1.\delta_m = \frac{\partial A}{\partial P_m} \quad \text{.........(17H)}$$

which is Castigliano's law and means that "the path δ_m of a load P_m is equal to the partial differential derivative of the actual work of the frame with respect to the load P_m ".

It is thus seen that Castigliano's law represents an equivalent for Mohr's work equation except that Castigliano arrives at his result by a more circuitous process.

For solid web structures involving direct stress and bending, Castigliano's equation takes the form.

$$1.\delta_m = \frac{1}{EF}\int \frac{N \partial N}{\partial P_m} dx + \frac{1}{EI}\int \frac{M \partial M}{\partial P_m} dx., \quad \text{.........(17K)}$$

where N and M are effects produced by any actual case of loading P_m

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Mohr's law, leading to a more direct solution of deflection problems, will generally be employed in preference to Castigliano, though the latter did receive earlier attention of American writers due to English translations of the Italian text.

Chap.IV. Statically Indeterminate Structures.

Art 18 Indeterminate Frames by Mohr's Work Equation.

It is now proposed to show the manner in which Mohr's work equation may be employed to find the stresses and reactions in any structure involving redundancy.

The application will be made first to frames, loaded with any system of loads P, concentrated at the several pin points of the frame.

In Art 11 it was shown that any problem involving redundancy could be reduced to a principal system which must be statically determinate, and then assign the redundants to the principal system as conventional loadings along with the external forces and reactions.

For each redundant so disposed or assigned, Mohr's work equation will give a means of finding the displacement of its point of application. Hence according to definitions given in Art.11, for redundants X_a, X_b, X_c, etc., and displacements in their points of application δ_a, δ_b, δ_c, ...etc., respectively, Mohr's work equation (14D) can be applied once for each redundant. Introducing the value $\varrho = l/EF$ for each member of the frame, these equations are

$$\left.\begin{aligned} \delta_a &= \Sigma S_a \frac{Sl}{EF} = \Sigma S_a S\varrho \\ \delta_b &= \Sigma S_b \frac{Sl}{EF} = \Sigma S_b S\varrho \\ \delta_c &= \Sigma S_c \frac{Sl}{EF} = \Sigma S_c S\varrho \end{aligned}\right\} \text{.........} \quad (18A)$$

etc., for any number of redundants, neglecting temperature effects and abutment displacements.

In these equations the value of S is represented by equation (11A), for any case of redundancy, as

$$S = S_o - S_a X_a - S_b X_b - S_c X_c - etc., \text{......................} \quad (18B)$$

and this value introduced into equations (18A) gives as many elasticity equations as there are redundants as follows:-

$$\left.\begin{aligned} \delta_a &= \Sigma S_a S\varrho = \Sigma S_a S_o \varrho - X_a \Sigma S_a^2 \varrho - X_b \Sigma S_a S_b \varrho - X_c \Sigma S_a S_c \varrho, etc. \\ \delta_b &= \Sigma S_b S\varrho = \Sigma S_b S_o \varrho - X_a \Sigma S_b S_a \varrho - X_b \Sigma S_b^2 \varrho - X_c \Sigma S_b S_c \varrho, etc. \\ \delta_c &= \Sigma S_c S\varrho = \Sigma S_c S_o \varrho - X_a \Sigma S_c S_a \varrho - X_b \Sigma S_c S_b \varrho - X_c \Sigma S_c^2 \varrho, etc. \end{aligned}\right\} \text{.......} \quad (18C)$$

wherein the summations include only the members of the principal system.

Now $\delta_a, \delta_b, \delta_c$, etc, being changes in lengths of redundant members of some kind, these may be evaluated in terms of their lengths, areas and stresses (or reactions) and become :-

$$\delta_a = \frac{X_a l_a}{E F_a}, \quad \delta_b = \frac{X_b l_b}{E F_b}, \quad \delta_c = \frac{X_c l_c}{E F_c}, \text{ etc.}, \quad \ldots\ldots\ldots\ldots\ldots (18D)$$

For external redundancy with immovable supports, $\delta_a = \delta_b = \delta_c = 0$.

Hence, all the terms in equations (18C) are now known, except the three redundant forces X_a, X_b, and X_c, and having three elasticity equations involving only three unknowns, the latter may be found by solving equations (18C) for simultaneous values of the X's, with the aid of equations (18D).

When the X's are found for a certain case of loading, then equation (18B) will give the stress in any member.

To include temperature effects or abutment displacements it is best to evaluate these separately with the use of the general work equation (14C).

Art. 19. Indeterminate Solid Web Structures by Mohr's Work Equation.

The procedure is similar to the one just given for frames except that Mohr's work equation (14E) is employed together with the general moment equation (11A). Since the effect due to axial or direct stress is usually neglible, we will consider only moments for the present.

As for frames, so also for solid web structures involving redundancy, we must first reduce the problem to a principal system and assign the redundants as external forces, moments or reactions, and conduct the analysis for the principal system selected. We will proceed on the basis of three redundants X_a, X_b, and X_c.

The work equations for the displacements of the points of application of the redundants are then according to equation (14E) :-

$$\left.\begin{aligned} 1\cdot\delta_a &= \frac{1}{EI}\int M M_a\,dx \\ 1\cdot\delta_b &= \frac{1}{EI}\int M M_b\,dx \\ 1\cdot\delta_c &= \frac{1}{EI}\int M M_c\,dx \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (19A)$$

wherein the moment M is the moment at any point of the principal system due to loads P and redundants X acting together, while the moments M_a, M_b and M_c are those moments in the principal system, respectively for the conventional loadings $X_a = 1$, $X_b = 1$ and $X_c = 1$.

Now the general moment equation (11A), including the effects of the redundants, is :-

$$M = M_o - M_a X_a - M_b X_b - M_c X_c \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (19B)$$

wherein M_o = the moment at any point of the principal system due to the actual loads P.

Substituting this value of M into equations (19A) the final elasticity equations are obtained as follows :-

$$\left.\begin{aligned} 1\cdot\delta_a &= \frac{1}{EI}\int M_o M_a\,dx - \frac{X_a}{EI}\int M_a^2\,dx - \frac{X_b}{EI}\int M_a M_b\,dx - \frac{X_c}{EI}\int M_a M_c\,dx \\ 1\cdot\delta_b &= \frac{1}{EI}\int M_o M_b\,dx - \frac{X_a}{EI}\int M_b M_a\,dx - \frac{X_b}{EI}\int M_b^2\,dx - \frac{X_c}{EI}\int M_b M_c\,dx \\ 1\cdot\delta_c &= \frac{1}{EI}\int M_o M_c\,dx - \frac{X_a}{EI}\int M_c M_a\,dx - \frac{X_b}{EI}\int M_c M_b\,dx - \frac{X_c}{EI}\int M_c^2\,dx \end{aligned}\right\} \quad \ldots\ldots (19C)$$

In these equations everything can be determined by pure statics except the redundants X which must be found by solving the three equations (19C)

The displacements δ_a, δ_b and δ_c become zero when they refer to the displacements of fixed supports and the redundants are external. If the supports are subject to displacements, then appropriate values must be found or assigned from known conditions of the problem in hand.

The evaluation of the integrals is accomplished in a very simple manner by substitution formulas which will be explained later in connection with practical problems. See Art. 26.

After the redundants are found from equations (19C), the moment at any point of the structure may be found from equation (19B), or any reaction may be evaluated from :-

$$R = R_o - R_a X_a - R_b X_b - R_c X_c \quad \text{.......................... (19D)}$$

In the above case three redundants were assumed to exist, but the number of equations and the terms in each may be extended by similarity to any number of redundants. Also the above expressions can be reduced at will to cover one or two redundants simply by omitting the terms which are not needed.

Art.20. Indeterminate structures by Maxwell's law.

The application of Maxwell's law of reciprocal displacements to any structure involving redundancy will now be given. This necessitates a transformation of equations (18C) in which the summations are now expressed in terms of elastic deformations of the structure instead of the stresses in the members.

In equations (18C) each term consists of the summation of the products of two stresses, which according to Mohr's work equation will represent some displacement value δ. It is, therefore, desirable to adopt some standard nomenclature which will always be understandable. This will consist of double subscript-bearing δ's, in which the first letter will always refer to the point at which the displacement occurs, and the second letter will always refer to the cause, or conventional loading producing the displacement.
For example let :-

δ_{ma} = the displacement of the point of application m of any load P_m, in the direction of this load, when the principal system is loaded only with the conventional load $X_a = 1$.

δ_{mb} = the displacement of the same point m for the conventional loading $X_b = 1$.

δ_{mc} = the same for a cause or condition $X_c = 1$.

δ_{aa} = the displacement of the point of application a of the redundant X_a in the direction of X_a, when the principal system is loaded only with $X_a = 1$.

δ_{ab} = a similar displacement of the point a due to the conventional loading $X_b = 1$.

δ_{ac} = a similar displacement of point a for $X_c = 1$, etc, for any other cases.

Mohr's work equation (14D) then gives the following values :-

$$\Sigma S_o S_a \rho = \Sigma P_m \delta_{ma} \; ; \quad \Sigma S_o S_b \rho = \Sigma P_m \delta_{mb} \; ; \quad \Sigma S_o S_c \rho = \Sigma P_m \delta_{mc}$$

$$\Sigma S_a^2 \rho = \delta_{aa} \; ; \; \Sigma S_b^2 \rho = \delta_{bb} \; ; \; \Sigma S_c^2 \rho = \delta_{cc} \; ; \; \Sigma S_a S_b \rho = \delta_{ab} \; ;$$

$$\Sigma S_a S_c \rho = \delta_{ac} \; ; \quad \Sigma S_c S_a \rho = \delta_{ca} \; ; \; \Sigma S_b S_a \rho = \delta_{ba} \; ; \; \Sigma S_b S_c \rho = \delta_{bc}$$

etc.,for other similar values. Hence the elasticity equations (18C) will assume the following form :-

$$\left.\begin{aligned} \delta_a &= \Sigma P_m \delta_{ma} - X_a \delta_{aa} - X_b \delta_{ab} - X_c \delta_{ac} \\ \delta_b &= \Sigma P_m \delta_{mb} - X_a \delta_{ba} - X_b \delta_{bb} - X_c \delta_{bc} \\ \delta_c &= \Sigma P_m \delta_{mc} - X_a \delta_{ca} - X_b \delta_{cb} - X_c \delta_{cc} \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (20\text{ A})$$

It should be noted that according to Maxwell's law , $\delta_{ab} = \delta_{ba}$; $\delta_{ac} = \delta_{ca}$; $\delta_{bc} = \delta_{cb}$; $\delta_{ma} = \delta_{am}$; etc.

Hence all the deflections in the first equation (20A) can be obtained from a deflection diagram drawn for the principal system with a single load $X_a = 1$. Those of the second equation for a similar loading of $X_b = 1$, and those of the third equation for $X_c = 1$. Again $\delta_a = \delta_b = \delta_c = 0$ for rigid supports.

Therefore, any method of obtaining these deflections, preferably by Williot-Mohr displacement diagrams or by area moments will make this method applicable for the solution of the redundants X. After these are found, then equation (11A) will furnish the means for finding stresses, reactions or moments in any structure with redundancy.

It should also be noted that equations (20A) apply to any structure whether framed or solid web.

Chapter V. Deflections.

Art.21. The variety of deflection problems encountered in bridge and structural designs is so great that we can scarcely do more than describe the methods employed in the more involved problems, and then concentrate on the class of problems with which we are more particularly concerned in the field of buildings.

When dealing with larger bridge problems and complicated loadings, the simplest and most complete solution for displacements of all pin points is obtained by the graphic method known as a Williot-Mohr diagram first presented by the author in a paper read before the Detroit Engineering Society in 1894, and published in the Jour. Assn. Eng. Soc.

Another method possessing greater accuracy and especially useful for stress computations in large framed bridges involving redundancy, is the semi-graphic method of elastic weights contributed by Professor Mohr in 1875.

All these have received exhaustive treatment in the author's treatise " The Kinetic Theory of Engineering Structures ", and will not receive further attention here.

The method of area moments affording either a graphic or analytic solution, was first developed by Professor Mohr in 1868. This is a very useful general method for solid web structures and should be thoroughly understood.

The method is especially useful when the entire elastic curve is desired for complex loading and variable moment of inertia.

The most comprehensive general solution of deflection problems pertaining to solid web structures is by Mohr's work equation (14E) and eq. (14F). The same law expressed by equation (14D) for frames, gives the deflection of a single point only.

Art. 22. Deflections of Beams by Area Moments. (Mohr. 1868.)

The deflection problem which it is proposed to solve may be stated as follows :-

For any straight beam simply supported on two supports and loaded in any manner by loads normal to the beam axis, to find the deflection δ_m at any point m of the beam, in a direction normal to the beam axis.

The solution is accomplished by drawing a moment diagram for the given case of loading, then treating this moment area as a load area (moment ordinates in kip-feet x lengths in feet) and finding the moment of the moment area load about the point m where the deflection is to be found. This moment divided by E I will be the deflection δ_m provided E be taken in kips per square foot, and I in ft^4 in conformity with the units chosen for moments and lengths.

Problem 22A.

The solution of a problem will best illustrate the method and we will take the simple beam with a uniform load p per foot to find δ_m at the center .

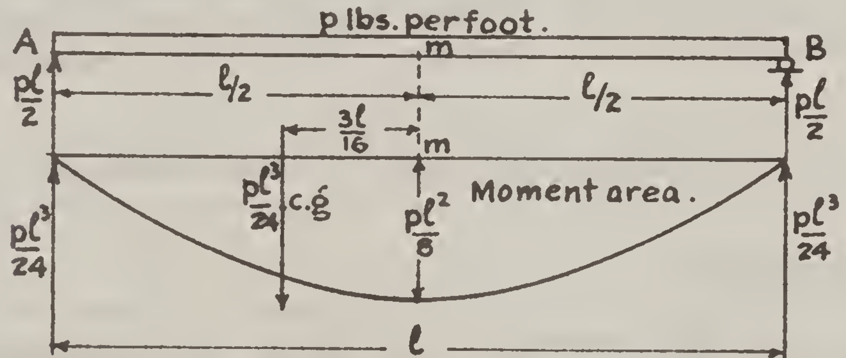


Fig. 22A.

The moment area for the uniform load p is a parabola with the middle ordinate at m equal to $\frac{p\,\ell^2}{8}$

The area of this parabola is equal to 2/3 times the length ℓ in feet, times the middle ordinate $\frac{p\,\ell^2}{8}$ in kip-feet.

$$= \frac{2}{3} \times \frac{p\,\ell^3}{8} = \frac{2p\ell^3}{24}.$$

Hence the end reactions A and B for the moment area will be equal to $\frac{p\,\ell^3}{24}$, and the resultant of the half area of equal weight will act through the center of gravity of the half area at $\frac{3\,\ell}{16}$ to the right or left of the center m.

Hence ,-

$$M_m = \frac{p\ell^3}{24}\left(\frac{\ell}{2} - \frac{3\ell}{16}\right) = \frac{5}{384}p\ell^4, \quad \text{and} \quad \delta_m = \frac{5p\ell^4}{384EI} .$$

Should any portion of the moment area be negative, then the moment of the negative moment area must be negative with respect to the moment of the positive area.

Problem 22 B. A simple beam with concentrated load P_m to find the deflection δ_m under the load.

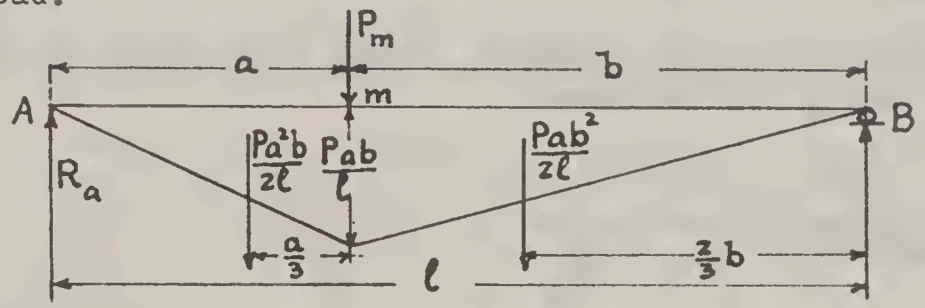


Fig.22B.

$$R_a = \frac{1}{\ell}\left[\frac{Pa^2b}{2\ell}\left(b+\frac{a}{3}\right) + \frac{Pab^2}{2\ell}\times\frac{2b}{3}\right] = \frac{1}{\ell}\left[\frac{Pa^2b^2}{2\ell} + \frac{Pa^3b}{6\ell} + \frac{Pab^3}{3\ell}\right] = \frac{Pab}{6\ell^2}\left[2b^2+3ab+a^2\right]$$

$$= \frac{Pab}{6\ell^2}\left[b\ell+\ell^2\right] = \frac{Pab}{6\ell}\left[b+\ell\right] = \frac{Pab}{6\ell}\left[a+2b\right].$$

$$M_m = aR_a - \frac{Pa^2b}{2\ell}\times\frac{a}{3} = \frac{Pa^3b}{6\ell} + \frac{Pa^2b^2}{3\ell} - \frac{Pa^3b}{6\ell} = \frac{Pa^2b^2}{3\ell}.$$

Hence $\delta_m = \frac{P_m}{3EI}\times\frac{a^2b^2}{\ell}$.

The solution of these problems is lengthier when dealing with lettered diminsions than with actual numerical cases.

Proof of the above analytic solution follows very simply from an application of Mohr's work equation (14E) which is :-

$$1.\delta_m = \frac{1}{EI}\int M M_a dx.$$

In this equation M_a is the moment at any point of a beam due to a unit load at m, but it is also equal to the moment at m due to a unit load at any point of the beam. Therefore, $M M_a dx$ represents the moment at m due to loads $M dx$ on any differential length dx and the expression $\int M M_a dx$ must represent the total moment at m. Hence this total moment divided by E I must equal $1.\delta_m$ according to Mohr's work equation (14E) thus establishing the basis for the method of area moments.

Art.23. Graphic solution of deflections by Area Moments. Mohr.

The problem is exactly the same as stated in Art. 22, and the solution is now accomplished by applying the well known graphic method of finding the moment of any set of forces w by means of a force polygon.and an equilibrium polygon. The forces in the present graphic solution will be the partial areas of a moment area and may be called elastic weights $w = Mdx$ for M in kipft., and dx in feet.

The properties of an equilibrium polygon are such that the moment at any point m of a beam, is the ordinate of the polygon at m, multiplied by the pole distance H employed in the force polygon.

But since the deflection δ_m is the moment of the elastic weights w divided by E I , therefore, if the force polygon be constructed for weights w with a pole H = E I, the ordinates to the equilibrium polygon would represent the actual deflections to the scale of lengths chosen in drawing the beam.

Now since these deflections would be too small for proper scaling, it is well to chose a pole $H = \frac{E\ I}{100}$, for which the deflection ordinates would be 100 times actual to the scale of lengths.

Problem 23 A. A simple beam on two supports, loaded with loads P, to find the deflection at any point m of the beam.

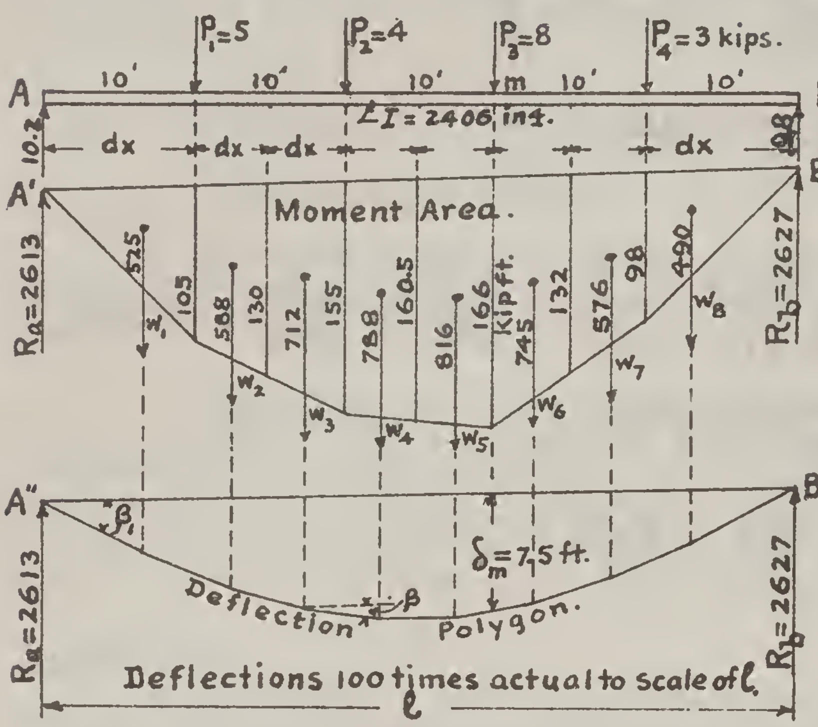
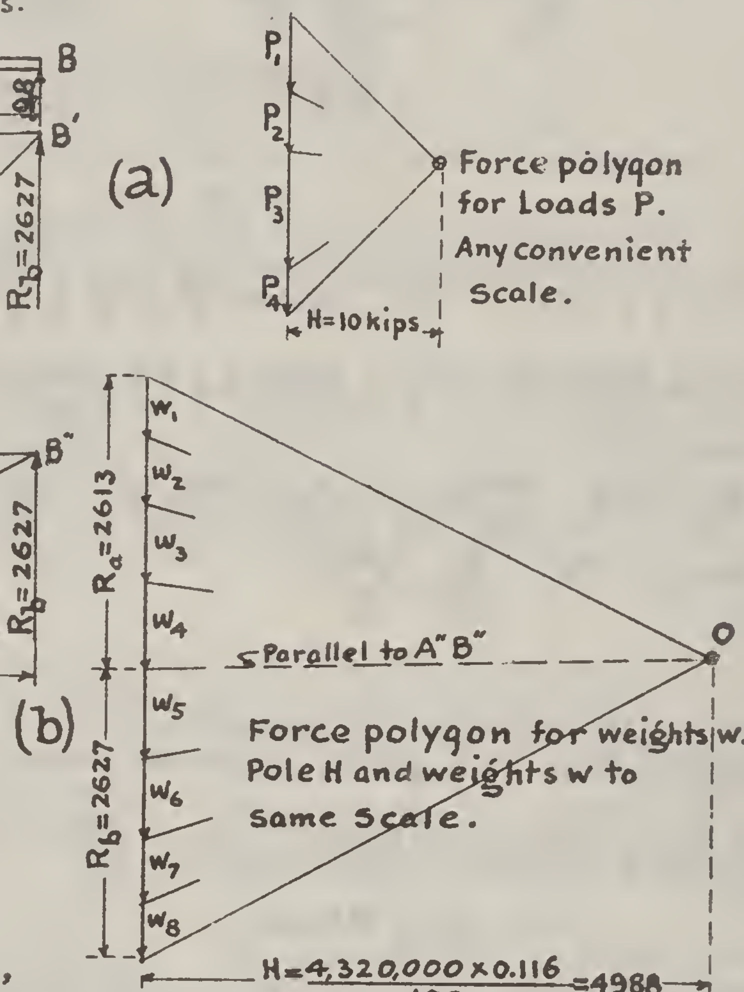


Fig. 23A.



Convenient units are as follows :-

Loads P in kips, lengths l, dx, feet,
Moments M in kip feet.

E = 30,000 x 144 = 4320,000 kips per square foot.

I in $ft.^4$ = $\frac{I \text{ in } In.^4}{12^4} = 0.116\ ft.^4$

The M moment diagram (**a**) drawn for the loads P, with pole 10 kips, gave moments 1/10 times actual to the scale of lengths, so that the actual values may be written on the moment area by direct scaling.

The M area is now suitably divided into partial lengths dx and the computed values of the w loads are then applied at the respective centers of gravity of the partial M areas.

The force and equilibrium polygons (**b**) are then drawn as indicated in Fig. 23 A, using a pole $H = \frac{E\ I}{100}$ thus obtaining a deflection polygon with ordinates δ_m which are 100 times actual to the scale of lengths employed for the span l. The entire elastic curve is thus found for the case of loads P as shown. *The actual value of δ_m under P_3 is found to be $1/100 \times 7.5$ ft. = 0.075 ft. = 0.90 inch.*

In cases where the moment of inertia is variable, it is best to choose the elastic weights $w = \frac{M\,dx}{I}$ and then employ a pole $H = \frac{E}{100}$ to obtain deflections 100 times actual to the scale of lengths.

No further proof of the graphic method is deemed necessary, since the principles involved are identical with those demonstrated for the analytic method in Art. 22.

Art.24. Slope of the elastic curve by Area Moments.

Since the slope of the M moment curve, or polygon, at any point of a beam is represented by the moment derivative, therefore $\frac{dM}{dx} = \tan\beta$.

But $dM/dx = Q$ = the shear at the point where the moment is taken. Also since the deflection δ_m is the moment of the elastic weights w about the point m, divided by E I, therefore, the shear of the elastic weights on one side of the point m, divided by E I, must equal the tangent of the slope angle to the elastic curve at the point m.

Also, the tangent of the elastic curve at any support must equal the end shear or reaction of the elastic weights w divided by E I.

In Fig.23A, the end reactions R_a and R_b of the elastic weights are found from the force polygon of these weights by drawing a line from the pole O, parallel to the closing line $A''B''$, thus dividing the total weights Σw into two portions representing the end reactions of the weights w. With R_a known, the shear at any point is easily found, giving :-

$$\tan\beta_1 = \frac{R_a}{EI} = \frac{2613}{498600} = 0.00525, \text{ and } \tan.\beta_m = \frac{Q_m}{EI}, \qquad \text{..........(24 A)}$$

This applies to either the graphic or analytic methods.

Art.25. Deflections of frames by Mohr's work equation.

Mohr's work equation (14 D) affords a means of solving any of the following four types of problems for any statically determinate frame and any case of externally applied loads, producing stresses S in the members of the given frame.

A. To find the displacement δ_m of any point m of the frame in any direction, by assigning a conventional load $P_m = 1$ in the particular direction in which the displacement is wanted. This represents the conventional loading of a point m

B. To find the relative displacement between any pair of points m and m_1 of the frame, by assigning a pair of conventional loads $P_m = 1$, acting in the direction m-m_1. This represents the loading of a pair of points m and m_1

C. To find the angular rotation of a line m-m_1 joining any two points m and m_1 of the frame, by assigning a conventional unit moment to the line m-m_1. We call this the conventional loading of a line m-m_1.

D. To find the change in the angle θ between a pair of lines m-m_1 and n-n_1 of the frame, by assigning a pair of conventional unit moments to the frame, one for each of the pair of lines. This represents the conventional loading of a pair of lines.

The four cases will be illustrated presently and each convention will be referred to as a loading, the δ_m of which will be the path or elastic displacement of the conventional loading.

If the supports undergo known displacements Δr as a result of the actual loads P, this effect must be considered in evaluating δ_m. Likewise temperature displacements may be found by applying temperature corrections to the Δl's of the members according to equation (8A). For the present, however, these effects will be neglected.

In the solution of problems it is very important to observe the signs of all stresses and Δl's . To avoid confusion, the positive sign will always be used to denote tension and elongation, while the negative sign will denote compression and contraction.

Problem 25 A. Displacement of a point m of any frame.

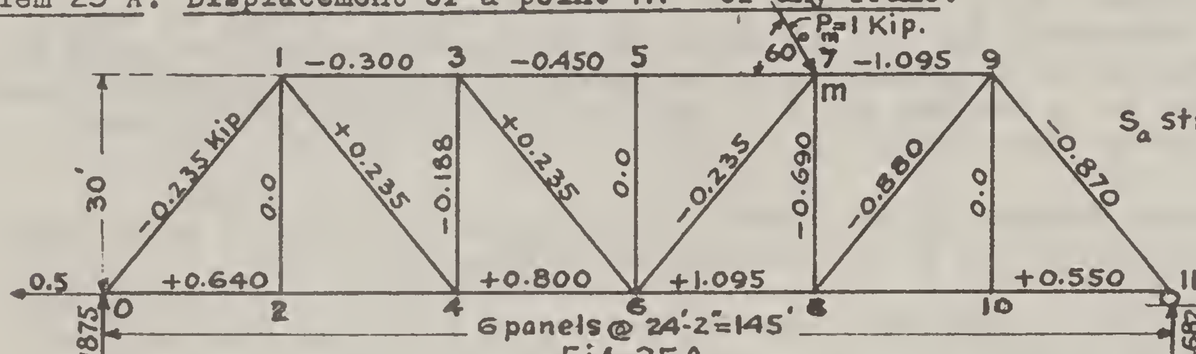


Fig.25A.

Given a simple truss on determinate supports, Fig. 25 A, find the displacement δ_m of the point m , in the direction of the unit load $P_m = 1$, when the frame is supporting some actual system of loads, producing stresses S, and changes $\Delta l = Sl/EF$ in the lengths of the members.

The solution is effected by evaluating Mohr's work equation (14 D) for the particular example given in Fig. 25 A, using the conventional loading $P_m = 1$, of a point m . The equation is :-

$$1.\ \delta_m = \Sigma S_a \frac{Sl}{EF} = \Sigma S_a \Delta l .$$

Compute the value Δl for each member, for the total stress S , length l and sectional area F of such member. Also, find the stress S_a in each member by drawing a Maxwell stress diagram for the conventional loading $P_m = 1$. Tabulate these values Δl and S_a as in Table 25 A, carefully observing the signs, + for tension and elongation, and then compute the products $S_a \Delta l$, the sum of which covering all the members, will give the desired deflection δ_m in the same length units as used for the Δl's .

TABLE 25A. SOLUTION OF PROBLEM 25A.

Mem	S	F	l	Δl	S_a	$S_a \Delta l$		Mem	S	F	l	Δl	S_a	$S_a \Delta l$	
	Kips	Sq.in	in.	in	Kips	+	−		Kips	Sq.in.	in	in.	Kips.	+	−
0-1	-245	33.5	462	-0.1126	-0.235	0.0265	-	1-2	+68	13.7	360	+0.0595	0.0	-	-
1-3	-252	26.5	290	-0.0920	-0.300	0.0276	-	9-10	+68	13.7	360	+0.0595	0.0	-	-
3-7	-284	29.4	580	-0.1864	-0.450	0.0840	-	3-4	-44	14.7	360	-0.0361	-0.188	0.0068	-
7-9	-252	26.5	290	-0.0920	-1.095	0.1008	-	7-8	-44	14.7	360	-0.0361	-0.690	0.0249	-
9-11	-245	33.5	462	-0.1126	-0.870	0.0980	-	5-6	-6.8	14.7	360	-0.0055	0.0	-	-
0-4	+158	15.9	580	+0.1925	+0.640	0.1230	-	1-4	+156	17.6	462	+0.1368	+0.235	0.0322	-
4-6	+252	26.9	580	+0.0906	+0.800	0.0725	-	8-9	+156	17.6	462	+0.1368	+0.880	0.1202	-
6-8	+252	26.9	580	+0.0906	+1.095	0.0993	-	3-6	+48	17.6	462	+0.0424	+0.235	0.0100	
8-11	+158	15.9	580	+0.1925	+0.550	0.1060	-	6-7	+48	17.6	462	+0.0424	-0.235	-	0.0100
					Forward	0.7377	o						$\Sigma S_a \Delta l =$	0.9318	0.0100

In the example all stresses are in kips, $E = 30,000$ kips per sq.in., lengths are in inches, and Cross-sections F in sq.inches.

Hence $\delta_m = +0.9318 - 0.0100 = +0.9218$ inch.

Problem 25 B. Relative displacement between any two points m and m_1 of any frame.

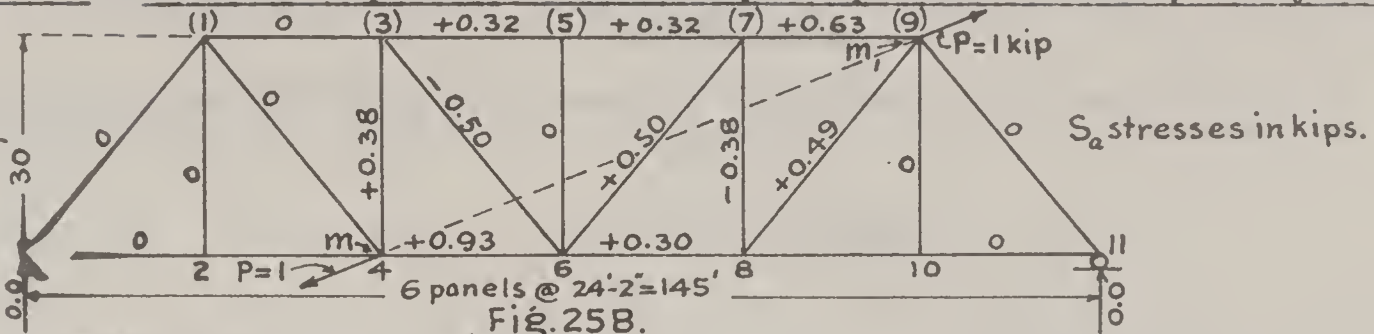


Fig.25B.

The example in Fig. 25 A is used again to illustrate the solution of this problem where δ_m is now the change in length of the distance m-m_1 which was produced by the changes $\Delta\ell$ in the lengths of the members. See above Table 25 A.

The conventional loading is that of a pair of points m and m_1 and consists of two equal and opposite loads $P=1$ applied at the two points and acting in the line m-m_1. In Fig,25 B, the unit loads are taken as acting away from each other on the presumption that $+\delta_m$ would be an elongation. Should the value of δ_m turn out as negative, then our assumption was wrong, and the arrows should point toward each other. This does not affect the solution of the problem, but merely corrects an erroneous assumption if such was made.

The stresses S_a for the two unit loads are again found from a Maxwell stress diagram, see Fig. 25 B, and the computations for δ_m are then carried out as in the previous Problem 25 A.
The answer is :-

$$\delta_m = +0.1922 - 0.1622 = +0.03 \text{ inch.}$$

Problem 25 C. Angular rotation of a line m-m_1, joining any two points m and m_1 of any frame.

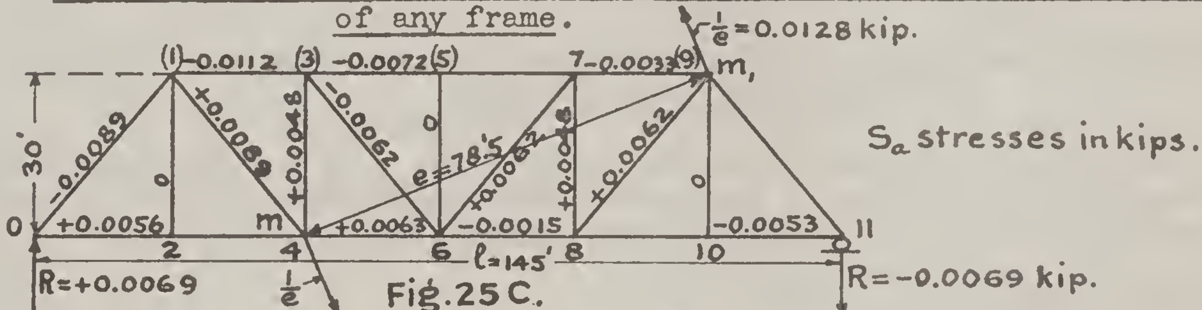


Fig.25C.

Again employing the example in Fig. 25 A, the rotation δ_m of the line m-m_1 will now be found.

We will assume the same changes $\Delta\ell$ in the lengths of the members and ascertain the rotation δ_m of the line joining the two points m and m_1.

The conventional loading is that of a line m-m_1 and consists of a unit moment produced by two forces $1/e$ acting in opposite directions through a lever arm e, and the rotation is assumed to be counterclockwise. Should the computed value of δ_m come out minus, then we would know that the rotation was actually clockwise instead of in the direction above assumed. The value of δ_m is expressed as the tangent of the angle of rotation.

In the above example, the S_a stresses as written on the members were obtained from a Maxwell diagram drawn for a moment $e \times \frac{1}{e} = 1$ kip ft...., and these stresses multiplied by the $\Delta\ell$'s of Table 25 A give the products $S_a \Delta\ell$, twelve times actual because the unit moment should have been 1 kip-inch instead of 1 kip ft.
The answer thus obtained is :-

$$12\,\delta_m = +0.00782 - 0.00258 = +0.00524, \text{ or } \delta_m = 0.00044,$$

which is the tangent of the angle of rotation of the line mm_1 .

Problem 25 D. Change in angle θ between a pair of lines AC and BC, of any frame.

The solution of this problem is illustrated on a three-hinged, framed arch, and involves the conventional loading of a pair of lines AC and BC, Fig. 25D.

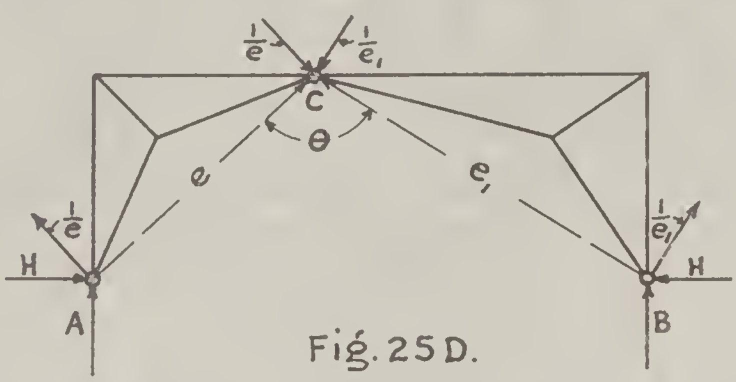


Fig. 25D.

The stresses S producing changes $\Delta \ell$ in the lengths of the members, which in turn cause the angular change δ_m in the angle θ, are found for any given case of loading and tabulated as in Table 25 A.

The conventional loading consists of a unit moment applied to each of the two lines, and the directions of rotation are so chosen that the angular change δ_m in the angle θ, is a positive increase expressed as the tangent of a small angle. The stresses S_a, due to the conventional loading indicated in Fig. 25 D, are now found by drawing a stress diagram or by computation, and these stresses added to the tabulated values $\Delta \ell$ for all the members of the structure, furnish the means for computing $1 \cdot \delta_m = \Sigma S_a \Delta \ell$.

The numerical solution being similar in every way to the previous examples, is not carried out in this case.

Art. 26. Deflection of solid web structures by Mohr's work equation, applicable to any principal system.

When dealing with solid web structures, the work equation takes a form which requires the integration of the products of two moments, over a distance for which the moment of inertia of the cross-section is constant.

Mohr's work equation (14D) is :-

$$1 \cdot \delta_m = \frac{1}{EI} \int_c^\ell M \, M_a \, dx$$

wherein :-

M = the moment of the externally applied loads P about any point m of a principal system.

M_a = the moment about any point m due to the conventional loading $P_m = 1$

δ_m = the deflection at the point m due to the applied loads P

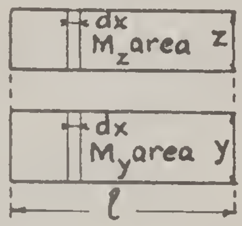
If it were not for the integrations involved in the solution of problems by means of the above equation, this would afford the simplest and most comprehensive general solution of deflection problems ever proposed.

Hence, the removal of the only deterrent factor, viz :- that of integration, will enhance the usefulness of Mohr's work equation and make it possible for anyone not familiar with the calculus, to solve quite easily any of the ordinary deflection problems and redundancy problems pertaining to solid web structures or rigid frames.

TABLE 26-A. EVALUATION OF THE EXPRESSIONS $\int_0^l M_z M_y dx$ and $\int_0^l M^2 dx$.

Moment of Inertia constant over length l.

1

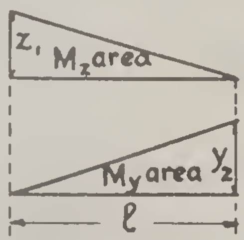


$\int_0^l M_z M_y dx = lzy.$

$\int_0^l M_z^2 dx = lz^2.$

$\int_0^l M_y^2 dx = ly^2.$

2

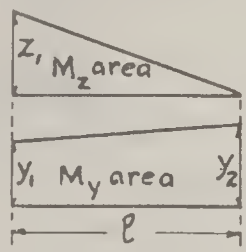


$\int_0^l M_z M_y dx = \frac{l}{6} z_1 y_2.$

$\int_0^l M_z^2 dx = \frac{l}{3} z_1^2$

$\int_0^l M_y^2 dx = \frac{l}{3} y_2^2$

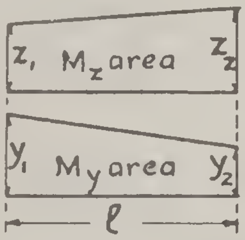
3



$\int_0^l M_z M_y dx = \frac{l z_1}{6}\left[2y_1 + y_2\right]$

$\int_0^l M_y^2 dx = \frac{l}{3}\left[y_1^2 + y_1 y_2 + y_2^2\right]$

4

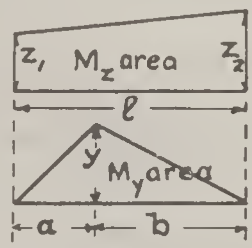


$\int_0^l M_z M_y dx = \frac{l}{6}\left[z_1(2y_1+y_2) + z_2(y_1+2y_2)\right]$

$\int_0^l M_z^2 dx = \frac{l}{3}\left[z_1^2 + z_1 z_2 + z_2^2\right].$

$\int_0^l M_y^2 dx = \frac{l}{3}\left[y_1^2 + y_1 y_2 + y_2^2\right]$

5



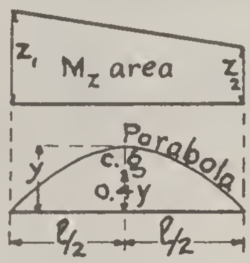
$\int_0^l M_z M_y dx = \frac{y}{6}\left[l(z_1+z_2) + bz_1 + az_2\right]$

when $z_1 = 0$, $\int M_z M_y dx = \frac{z_2 y}{6}(l+a)$

when $z_2 = 0$, $\int M_z M_y dx = \frac{z_1 y}{6}(l+b)$

$\int_0^l M_y^2 dx = \frac{l}{3} y^2.$

6

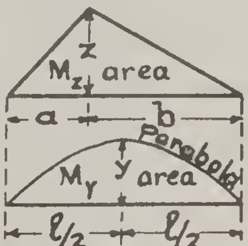


$\int_0^l M_z M_y dx = \frac{l}{3}(z_1 + z_2) y.$

$\int_0^l M_y^2 dx = \frac{8}{15} l y^2.$

M_y area $= \frac{2}{3} l y.$

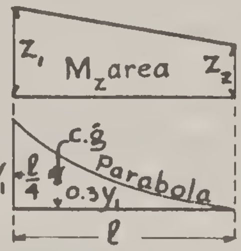
7



$\int_0^l M_z M_y dx = \frac{zy}{3}\left[l + \frac{ab}{l}\right].$

$\int_0^l M_y^2 dx = \frac{8}{15} l y^2.$

8

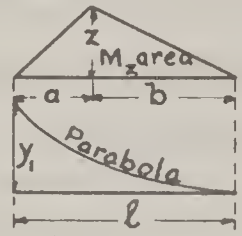


$\int_0^l M_z M_y dx = \frac{l y_1}{4}\left[z_1 - \frac{z_2}{3}\right].$

$\int_0^l M_y^2 dx = \frac{l}{5} y_1^2.$

M_y area $= \frac{l}{3} y_1$

9

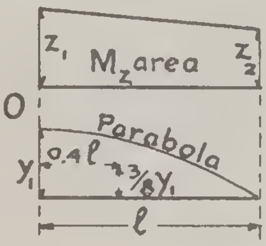


$\int_0^l M_z M_y dx = \frac{z}{2}\left[y_1\left(a + \frac{b}{2}\right) + \frac{a^3}{12}\right]$

$\int_0^l M_z^2 dx = \frac{l}{3} z^2.$

$\int_0^l M_y^2 dx = \frac{l}{5} y_1^2.$

10

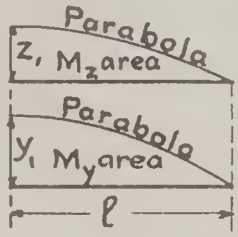


$\int_0^l M_z M_y dx = \frac{l y_1}{4}\left(\frac{5}{3} z_1 + z_2\right)$

$\int_0^l M_y^2 dx = \frac{8}{15} l y_1^2.$

M_y area $= \frac{2}{3} l y_1.$

11

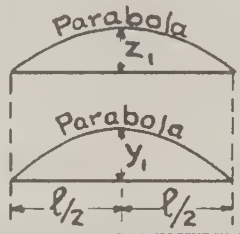


$\int_0^l M_z M_y dx = \frac{8}{15} l z_1 y_1.$

$\int_0^l M_y^2 dx = \frac{8}{15} l y_1^2$

$y = \frac{y_1}{l^2}(2lx - x^2).$

12

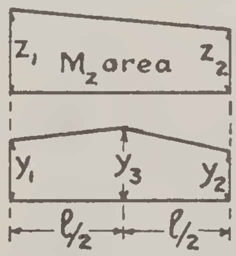


$\int_0^l M_z M_y dx = \frac{8}{15} l z_1 y_1.$

$\int_0^l M_y^2 dx = \frac{8}{15} l y_1^2.$

$y = \frac{4 y_1 x}{l^2}(l - x).$

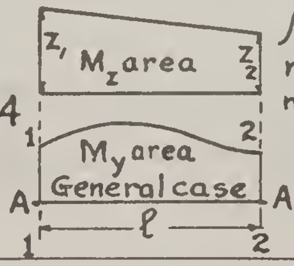
13



$\int_0^l M_z M_y dx = \frac{l}{24}\left[5(z_1 y_1 + z_2 y_2) + 6 y_3 (z_1 + z_2) + z_1 y_2 + z_2 y_1\right].$

To approximate a curve by three ordinates.

14



$\int M_z M_y dx = \frac{1}{l}\left[z_1 m_2 + z_2 m_1\right]$

m_1 = static mom. of M_y area about 1-1

m_2 = static mom. of M_y area about 2-2

$\int_0^l M_y^2 dx = 2m$, where m = static mom. of M_y area about A-A axis.

Note: In case the My and M_z moment diagrams have different lengths l, then the integral $\int M_z M_y dx$ can cover only that length which is common to both areas, because the moment ordinates of the shorter area are zero over that portion of the length which is not common to both moment areas.

The ordinates y and z are all positive in the above formulas, and whenever negative areas or ordinates occur, they must be introduced with negative signs. For a negative area, the integral $\int M^2 dx$ is a positive quantity.

David Molitor, C.E.
1927.

After some study the writer discovered that these integrations could be performed once for all, thus exhausting all possible combinations of moment diagrams which might be encountered.

The Table 26 A gives the evaluation of the integrals $\int_0^\ell M_z M_y \, dx$ and $\int_0^\ell M^2 dx$, for all possible moment diagrams and will prove quite valuable as a practical expedient in solving numerical problems.

The formulas may be called <u>substitution formulas</u>, and their use will be illustrated in Chapter <u>VI</u> by solving a number of problems.

CHAPTER 6. PROBLEMS BY MOHR'S WORK EQUATION.

Art. 27. Deflection Problems.

<u>Problem 27A</u>. Find the deflection at any point m of a simple beam, for the case of a uniform load p per foot of span, using the formula $1 \cdot \delta_m = \frac{1}{EI}\int_0^\ell M M_a \, dx$.

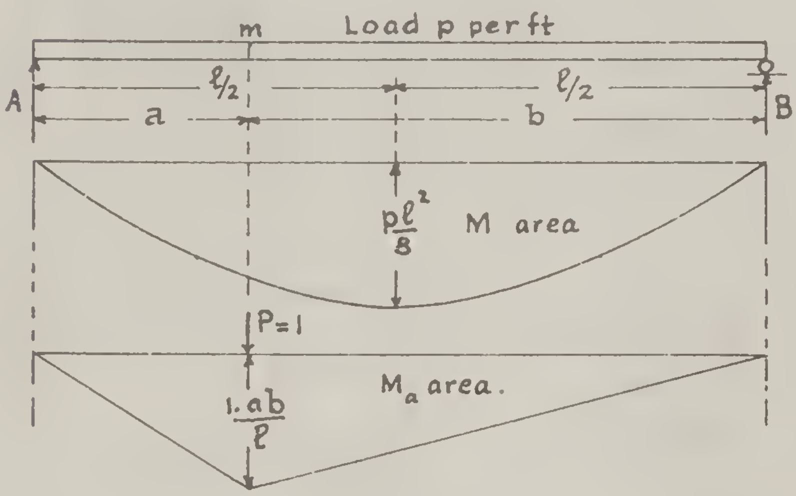


Fig. 27A.

<u>Solution</u>. Sketch the M area which is a parabola with middle ordinate $= \frac{p\ell^2}{8}$. Sketch the M_a area, which is a triangle with max. ordinate $= 1 \cdot \frac{ab}{\ell}$, Fig. 27A. Now employ the substitution Formula Table 26A, Case 7, making $y = \frac{p\ell^2}{8}$ and $z = \frac{ab}{\ell}$. Then we have directly

$$\int_0^\ell M_z M_y \, dx = \frac{zy}{3}\left[\ell + \frac{ab}{\ell}\right] = \frac{ab}{\ell} \times \frac{p\ell^2}{24}\left(\ell + \frac{ab}{\ell}\right)$$

$$\text{or } \delta_m = \frac{pab\ell}{24EI}\left(\frac{\ell^2 + ab}{\ell}\right) = \frac{pab}{24EI}\left(\ell^2 + ab\right).$$

This gives at once the elastic curve for the deflection δ_m at any point m. The maximum deflection at the center occurs when $a = b = \frac{\ell}{2}$, making $\delta_m = \frac{5p\ell^4}{384EI}$. Compare this with Problem 22A where the same case is solved by area-moments.

<u>Problem 27B</u>. Find the deflection δ_m at m under the concentrated load P_m acting on a simple beam AB.

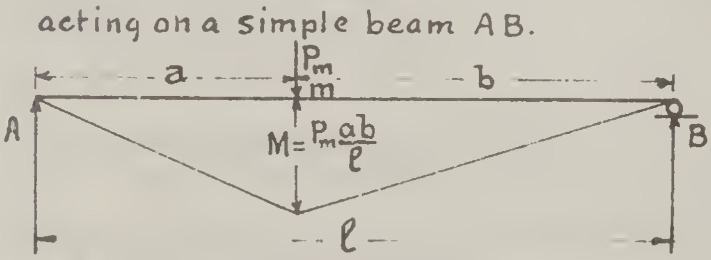
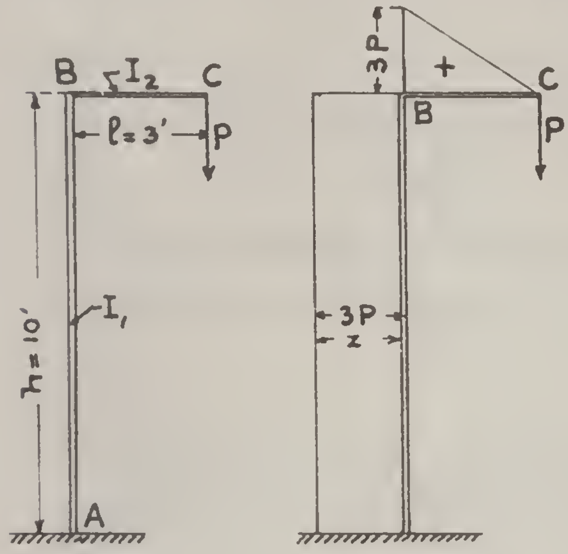


Fig. 27B.

Since the conventional load $P = 1$ now acts at the load point m, Eq. 14F is applicable giving $\delta_m = \frac{1}{P_m EI}\int M^2 dx$.

Fig. 27B shows the moment diagram with the moment ordinate $M = P_m \frac{ab}{\ell}$ under the load point m. Hence from Table 26A, Case 5, $\int_0^\ell M_y^2 dx = \frac{1}{3}y^2\ell = \frac{a^2b^2\ell}{3\ell^2}P_m^2$ giving $\delta_m = \frac{P_m}{3EI} \times \frac{a^2b^2}{\ell}$. Compare this with Problem 22B, solved by area-moments.

Problem 27C. Find the horizontal and the vertical deflections of the point C for the column AB with cantilever arm BC supporting a vertical load P at C.



Given Case M diagram.

Fig. 27C.

a. Solution for the vertical deflection δ_c of the point C.

Since the point C is the loaded point, and the required defl. is in the direction of P, we may apply the conventional load P and use Eq. 14F which is $\delta_c = \frac{1}{PEI}\int M^2 dx$.

The moment diagram for P is drawn in Fig. 27C, and the integral $\int M^2 dx$ is evaluated for the height h and the length ℓ of the cantilever arm.

From Table 26A, case 1, we obtain for the column:

$$\int_0^h M^2 dx = hz^2 = 10(3P)^2 = 90P^2.$$

Also from case 2 of same table, for the cantilever arm

$$\int_0^\ell M^2 dx = \frac{\ell}{3}z^2 = \frac{3}{3}(3P)^2 = 9P^2$$

Hence $\delta_c = \frac{1}{PEI}\int M^2 dx = \frac{1}{PE}\left(\frac{90P^2}{I_1} + \frac{9P^2}{I_2}\right) = \frac{P}{E}\left(\frac{90}{I_1} + \frac{9}{I_2}\right)$.

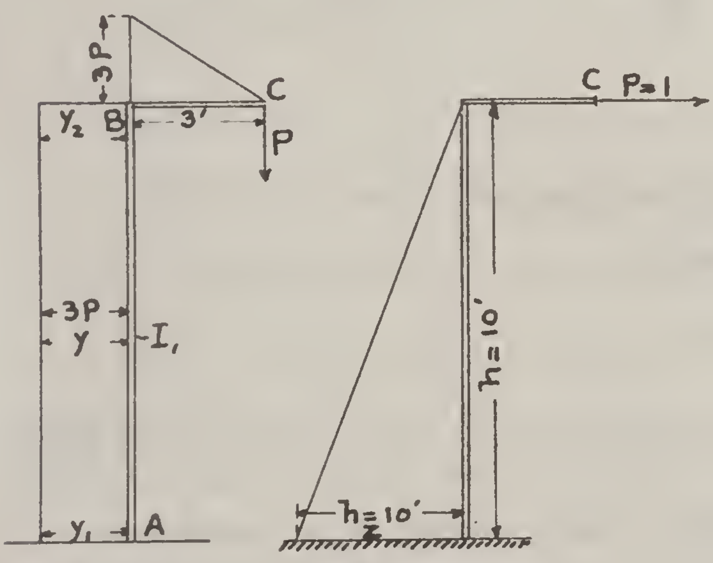
For $P = 4$ kips, $E = 30{,}000 \times 144 = 4{,}320{,}000$ kips per sq. ft. and $I_1 = I_2 = 12$ in$^4 = \frac{1}{12^3}$ ft^4 making $EI = 2500$ in kip foot units, then we have for the vertical deflection at C,

$$\delta_c = \frac{4 \times 99}{2500} = 0.1584 \text{ ft.} = 1.90 \text{ in.}$$

This deflection neglects the shortening in the column due to the direct stress P.

b. Solution for the horizontal deflection of the point C.

We must now apply a conventional load $P = 1$, horizontally at C, and employ Eq. 14d, which is $\delta_h = \frac{1}{EI}\int M M_a dx$.



M diagram M_a diagram.

The moment diagram for P is the same as before, but the M_a diagram for a horizontal load $P = 1$ at C, is a triangle with base $h = 10$ft.

Since the M_a area is zero on the cantilever, the integration $\int M M_a dx$ applies to the column only.

From Table 26A, case 3, we obtain the value

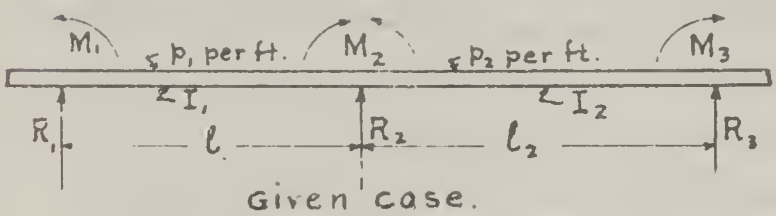
$\int M M_a dx = \frac{hz}{6}(2y_1 + y_2) = \frac{10 \times 10}{6}(6P + 3P) = 150P$,

making $\delta_h = \frac{150P}{EI_1}$, which for $P = 4$ kips and $EI = 2500$ as above, then

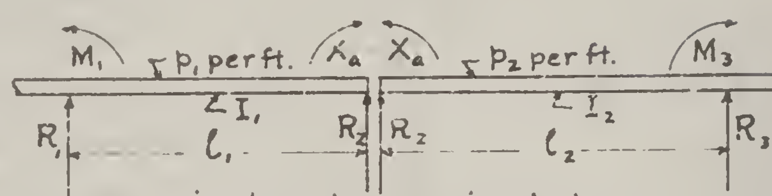
$$\delta_h = \frac{4 \times 150}{2500} = 0.24 \text{ ft.} = 2.88 \text{ in.}$$

Art. 28. The Three-Moment Equation and its Applications.

Problem 28A. Proof of the Three-Moment Equation, Uniform Loads, Variable I.



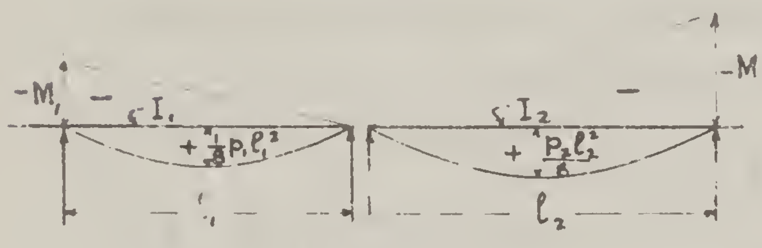
Given case.



Principal System - 2 simple beams.

M_1 and M_3 are regarded as known external moments and $M_2 = X_a$ is a redundant to be evaluated.

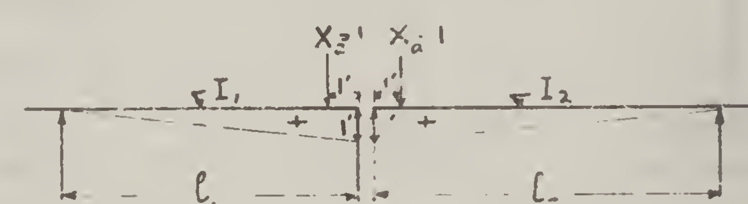
(a) (b)



Condition $X=0$, M_o areas.

The M_o area consists of 4 separate portions, 2-negative areas and 2-positive areas.

(c)



Condition $X_a = -1$, M_a areas.

The $X_a = 1$ moment is produced by a unit load acting 1 ft. from the end of each span.

(d)

The elasticity Equation for one redundant condition, according to Eq. 19C is:

$1.\delta_a = \frac{1}{EI}\int M_o M_a dx - \frac{X_a}{EI}\int M_a^2 dx = 0$ for an unyielding support R_2. E is constant and cancels.

Using the formulas Case 2 and Case 6 from Table 26A, we can now evaluate the two integrals, observing that $\int M_o M_a dx$ must be made to cover the two negative and the two positive M_o areas, combined with the M_a area of the same span. This gives at once,

$$\frac{1}{I}\int M_o M_a dx = \frac{l_1}{3I_1}\times 1 \times \frac{p_1 l_1^2}{8} + \frac{l_2}{3I_2}\times 1 \times \frac{p_2 l_2^2}{8} - \frac{l_1}{6I_1}M_1 - \frac{l_2}{6I_2}M_3$$

and

$$\frac{X_a}{I}\int M_a^2 dx = \frac{X_a}{3}\left(\frac{l_1}{I_1} + \frac{l_2}{I_2}\right).$$

These values substituted in the above elasticity equation give:

$$\frac{p_1 l_1^3}{24 I_1} + \frac{p_2 l_2^3}{24 I_2} - \frac{M_1 l_1}{6 I_1} - M_3\frac{l_2}{6I_2} - \frac{X_a}{3}\left(\frac{l_1}{I_1} + \frac{l_2}{I_2}\right) = 0 \quad \text{(1)}$$

Multiplying Eq. 1 by 6 and transposing, then

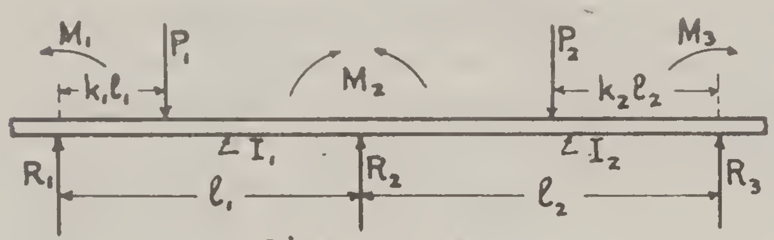
$$M_1\frac{l_1}{I_1} + 2X_a\left(\frac{l_1}{I_1} + \frac{l_2}{I_2}\right) + M_3\frac{l_2}{I_2} = \frac{p_1 l_1^3}{4I_1} + \frac{p_2 l_2^3}{4I_2} \quad \text{(28A)}$$

which is the Three-Moment Equation for Uniform Loads. The positive signs merely prove that $X_a = M_2$ was properly assigned in Fig. (b).

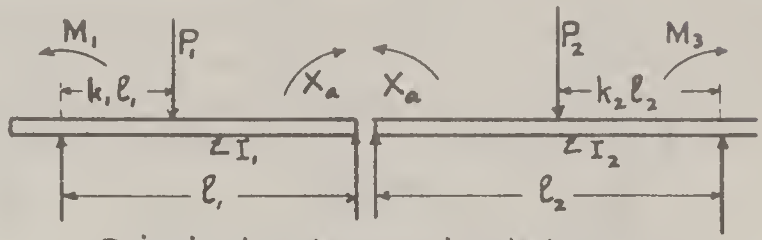
Knowing that the three terms of the left hand side of Eq. 28A represent negative moments, therefore, if this equation is to give the proper sign we must place the negative sign before the two right hand terms and obtain the final equation as:

$$M_1\frac{l_1}{I_1} + 2M_2\left(\frac{l_1}{I_1} + \frac{l_2}{I_2}\right) + M_3\frac{l_2}{I_2} = -\frac{p_1 l_1^3}{4I_1} - \frac{p_2 l_2^3}{4I_2} = -\Sigma\frac{pl^3}{4I} \quad \text{(28B)}$$

Problem 28B. Proof of the Three-Moment Equation, Concentrated Loads, Variable I.



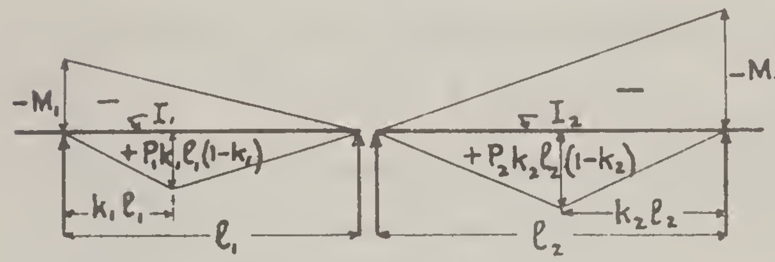
Given case



Principal System - 2 simple beams.

M_1 and M_3 are regarded as known external moments and $M_2 = X_a$ is a redundant to be evaluated.

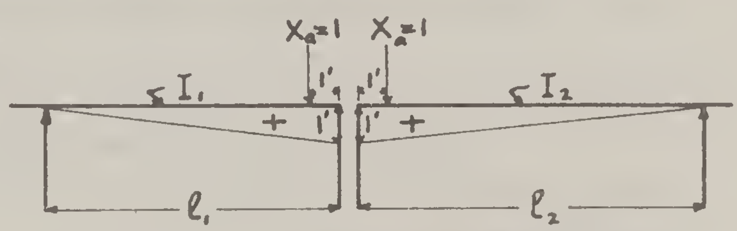
(a) (b)



Condition $X = 0$, M_o areas.

The M_o area consists of 4 separate portions, 2-negative areas and 2-positive areas.

(c)



Condition $X_a = -1$, M_a areas.

The $X_a = 1$ moment is produced by a unit load acting 1 ft. from the end of each span.

(d)

The elasticity Equation for one redundant condition, according to Eq. 19C is;

$1\cdot\delta_a = \frac{1}{EI}\int M_o M_a dx - \frac{X_a}{EI}\int M_a^2 dx = 0$ for an unyielding support R_2. E is constant and cancels.

Using the Formulas Case 2 and Case 5, from Table 26A, we can now evaluate the two integrals, observing that $\int M_o M_a dx$ must be made to cover the two negative and the two positive M_o areas, combined with the M_a area of the same span. This gives at once,

$$\frac{1}{I}\int M_o M_a dx = \frac{\ell_1}{6I_1} P_1 k_1(1-k_1)(\ell_1 + k_1\ell_1) + \frac{\ell_2}{6I_2} P_2 k_2(1-k_2)(\ell_2 + k_2\ell_2) - \frac{M_1\ell_1}{6I_1} - \frac{M_3\ell_2}{6I_2}$$

and

$$\frac{X_a}{I}\int M_a^2 dx = \frac{X_a}{3}\left(\frac{\ell_1}{I_1} + \frac{\ell_2}{I_2}\right).$$

These values substituted in the above elasticity equation, and noting $(1-k)(1+k) = 1-k^2$, give:

$$\frac{\ell_1^2}{6I_1} P_1 k_1(1-k_1^2) + \frac{\ell_2^2}{6I_2} P_2 k_2(1-k_2^2) - \frac{M_1\ell_1}{6I_1} - \frac{M_3\ell_2}{6I_2} - \frac{X_a}{3}\left(\frac{\ell_1}{I_1} + \frac{\ell_2}{I_2}\right) = 0 \text{ ----------- } (1)$$

Multiplying Eq. 1 by 6 and transposing, then

$$M_1\frac{\ell_1}{I_1} + 2X_a\left(\frac{\ell_1}{I_1} + \frac{\ell_2}{I_2}\right) + M_3\frac{\ell_2}{I_2} = P_1k_1(1-k_1^2)\frac{\ell_1^2}{I_1} + P_2k_2(1-k_2^2)\frac{\ell_2^2}{I_2} \text{ --------- } (28C)$$

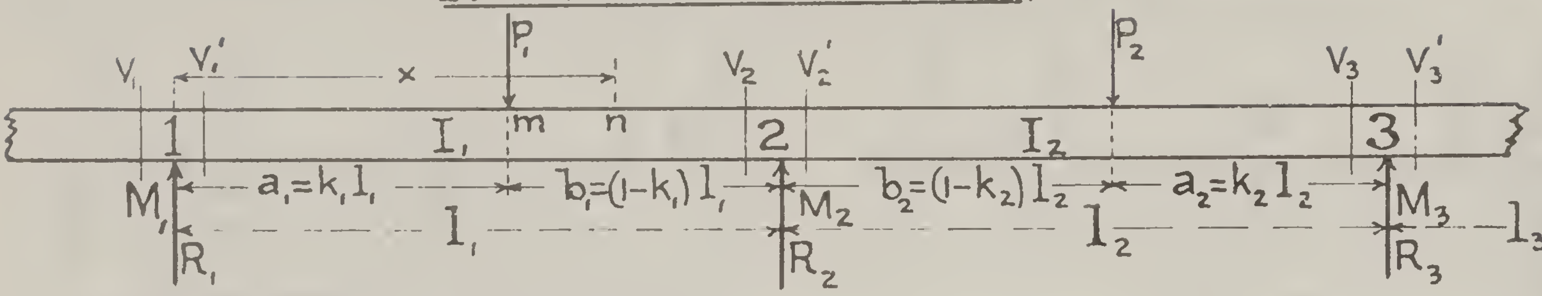
which is the familiar Three-Moment Equation for concentrated loads. The positive signs merely indicate that $X_a = M_2$ was properly assigned in Fig. (b).

Knowing that the three terms of the left hand side of Eq. 28C represent negative moments, therefore, if this equation is to give the proper sign we must place the negative sign before the two right hand terms and obtain the final equation as:

$$M_1\frac{\ell_1}{I_1} + 2M_2\left(\frac{\ell_1}{I_1} + \frac{\ell_2}{I_2}\right) + M_3\frac{\ell_2}{I_2} = -P_1k_1(1-k_1^2)\frac{\ell_1^2}{I_1} - P_2k_2(1-k_2^2)\frac{\ell_2^2}{I_2} \text{ ----- } (28D)$$

THREE MOMENT EQUATION.

I.- CONCENTRATED LOADS.



GENERAL MOMENT EQUATIONS.

$$M_1\frac{l_1}{I_1} + 2M_2\left(\frac{l_1}{I_1}+\frac{l_2}{I_2}\right) + M_3\frac{l_2}{I_2} = -P_1 l_1 k_1(1-k_1^2)\frac{l_1}{I_1} - P_2 l_2 k_2(1-k_2^2)\frac{l_2}{I_2} = -\Sigma P l k(1-k^2)\frac{l}{I} \quad \text{---(1)}$$

$$M_n = P_1k_1(l_1-x) - M_1\left(1-\frac{x}{l_1}\right) - M_2\frac{x}{l_1} = P_1\left(a_1-\frac{x}{l_1}\right) - M_1\left(1-\frac{x}{l_1}\right) - M_2\frac{x}{l_1} \quad \text{---(2)}$$

$$M_m = P_1 l_1 k_1(1-k_1) - M_1(1-k_1) - M_2k_1 = \frac{1}{l_1}\left[P_1a_1b_1 - M_1b_1 - M_2a_1\right] \quad \text{---(3)}$$

Positive moments produce tension on the lower side of beam and compr. on upper side.

Negative moments produce tension on the upper side of beam and compr. on lower side.

SHEARS. First span.

$$V_1' = \frac{M_2-M_1}{l_1} + P_1(1-k_1) \quad \text{for points from 1 to m. ---(4)}$$

$$V_2 = V_1' - P_1 = \frac{M_2-M_1}{l_1} - P_1k_1 \quad \text{for points from m to 2. ---(5)}$$

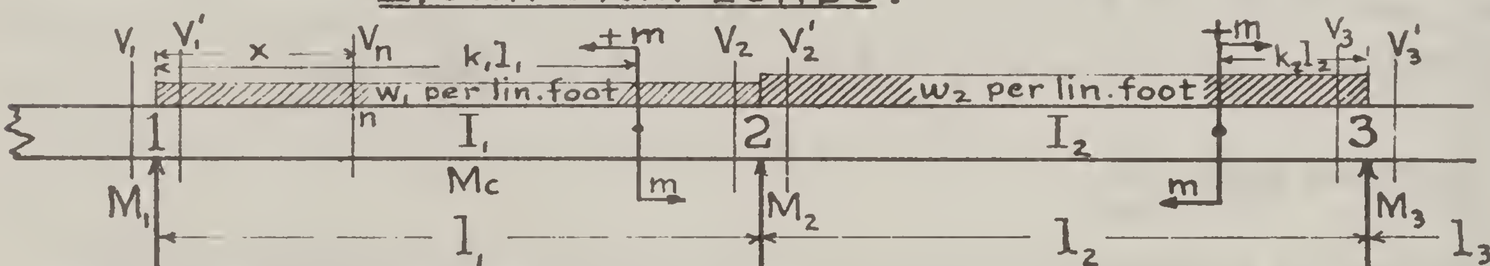
SHEARS. Second span.

$$V_2' = \frac{M_3-M_2}{l_2} + P_2k_2 \quad \text{for points from 2 to } P_2. \qquad V_3 = V_2' - P_2 \quad \text{---(6)}$$

M_1, M_2 and M_3 are substituted with their proper signs.

REACTIONS. $R_1 = V_1+V_1'$; $R = V_2+V_2'$ and $R_3 = V_3+V_3'$,etc. ---(7)

II.- UNIFORM LOADS.



GENERAL MOMENT EQUATIONS.

$$M_1\frac{l_1}{I_1} + 2M_2\left(\frac{l_1}{I_1}+\frac{l_2}{I_2}\right) + M_3\frac{l_2}{I_2} = -\frac{w_1l_1^2}{4}\cdot\frac{l_1}{I_1} - \frac{w_2l_2^2}{4}\cdot\frac{l_2}{I_2} = -\Sigma\frac{wl^2}{4}\cdot\frac{l}{I} \quad \text{---(8)}$$

$$M_n = M_1\left(1-\frac{x}{l_1}\right) + M_2\frac{x}{l_1} + \frac{w_1x}{2}(l_1-x) = M_1 + V_1'x - \frac{w_1x^2}{2}, \text{ max. when } x = \frac{V_1'}{w_1} \quad \text{---(9)}$$

$$M_c = \frac{w_1l_1^2}{8} + \frac{M_1+M_2}{2}, \text{ using proper signs for } M_1 \text{ and } M_2 \text{. ---(10)}$$

SHEARS: $V_1' = \frac{M_2-M_1}{l_1} + \frac{w_1l_1}{2}$; $V_2 = V_1' - w_1l_1$; $V_2' = \frac{M_3-M_2}{l_2} + \frac{w_2l_2}{2}$; $V_3 = V_2' - w_2l_2$ ---(11)

$V_n = \frac{M_2-M_1}{l_1} + w_1\left(\frac{l_1}{2}-x\right)$ which becomes zero for $x = \frac{V_1'}{w_1}$ ---(12)

REACTIONS as for concentrated loads.

THREE MOMENT EQUATION FOR ALL LOAD EFFECTS COMBINED:

$$M_1\frac{l_1}{I_1} + 2M_2\left(\frac{l_1}{I_1}+\frac{l_2}{I_2}\right) + M_3\frac{l_2}{I_2} = -\Sigma Plk(1-k^2)\frac{l}{I} - \Sigma\frac{wl^2}{4}\cdot\frac{l}{I} - \Sigma m(3k^2-1)\frac{l}{I} \quad \text{---(13)}$$

The m moments are positive when acting clockwise in the right hand span, and when acting counterclockwise in the lefthand span as shown.

D. Molitor, Apr. 1926

Problems 28C. Applications of the Three-Moment Equation – Constant I.

General Equation-Concentrated Loads ÷ $M_1\ell_1 + 2M_2(\ell_1+\ell_2) + M_3\ell_2 = -\Sigma Pk\ell^2(1-k^2)$ – – – – – – (1)

General Equation- Uniform Loads – $M_1\ell_1 + 2M_2(\ell_1+\ell_2) + M_3\ell_2 = -\Sigma \frac{p\ell^3}{4}$ – – – – – – – – – – (2)

Problem A. Beam fixed at one end, simply supported at the other.

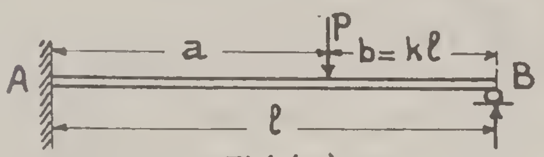


Fig (a)

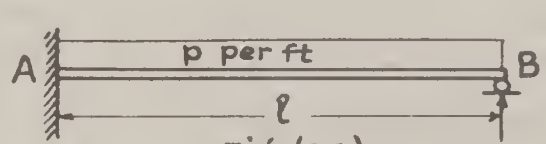


Fig (2a)

In the general Eqs. make $\ell_1 = 0$ and $M_3 = 0$ and write the new equation for M_2 at A.

Then from Fig(a), $2M_A(\ell+0) = -Pk\ell^2(1-k^2)$ giving $M_A = -\frac{Pk\ell}{2}(1-k^2) = -\frac{Pab}{2\ell^2}(\ell+b)$

Also from Fig(2a), $2M_A(\ell+0) = -\frac{p\ell^3}{4}$ giving $M_A = -\frac{p\ell^2}{8}$

Problem B. Beam on 3 Supports, not restrained at the outer supports.

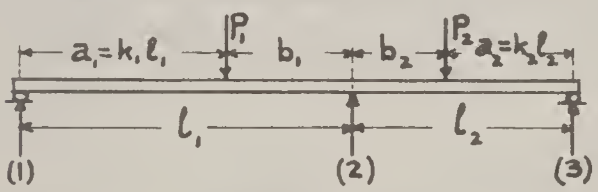


Fig (b)

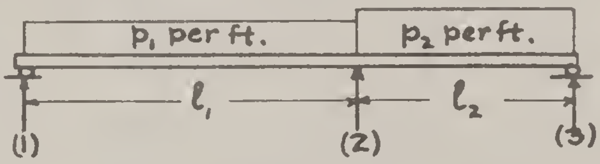


Fig (2b)

In the general Equations make $M_1 = M_3 = 0$, and apply with middle term at (2) as written:

Then $2M_2(\ell_1+\ell_2) = -P_1k_1\ell_1^2(1-k_1^2) - P_2k_2\ell_2^2(1-k_2^2)$

giving $M_2 = -\frac{P_1k_1\ell_1^2(1-k_1^2) + P_2k_2\ell_2^2(1-k_2^2)}{2(\ell_1+\ell_2)}$

Also for Fig(2b), $2M_2(\ell_1+\ell_2) = -\frac{p_1\ell_1^3}{4} - \frac{p_2\ell_2^3}{4}$.

giving $M_2 = -\frac{p_1\ell_1^3 + p_2\ell_2^3}{8(\ell_1+\ell_2)}$

Problem C. Beam fixed at both ends.

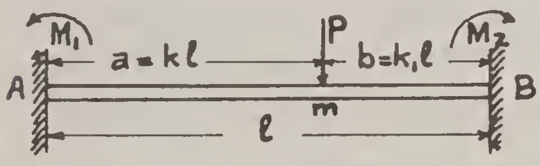


Fig (C)

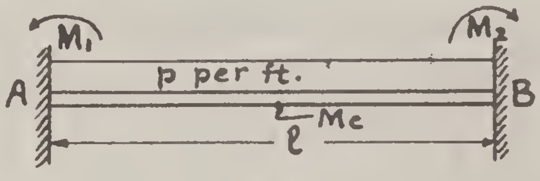


Fig (2C)

Apply Eq. 1, first with middle term at A for $\ell_1 = 0$, and again with middle term at B for $\ell_2 = 0$, This gives the following 2 equations:

$$2M_1(\ell+0) + M_2\ell = -Pk_1\ell^2(1-k_1^2) = -\frac{Pab}{\ell}(\ell+b)$$

$$M_1\ell + 2M_2(\ell+0) = -Pk\ell^2(1-k^2) = -\frac{Pab}{\ell}(\ell+a)$$

These are solved as follows:

$$M_1 + 0.5M_2 = -\frac{Pab}{2\ell^2}(\ell+b)$$

$$M_1 + 2.0M_2 = -\frac{Pab}{\ell^2}(\ell+a)$$

Subtracting:

$$1.5M_2 = -\frac{Pab}{2\ell^2}(2\ell+2a-\ell-b)$$

giving $M_2 = -\frac{Pab}{3\ell^2}(\ell-b+2a) = -\frac{Pa^2b}{\ell^2}$

and $M_1 = -\frac{Pab}{3\ell^2}(\ell-a+2b) = -\frac{Pab^2}{\ell^2}$

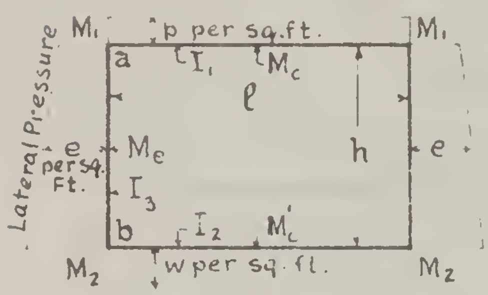
For uniform load $M_1 = M_2$ so that we apply Eq(2) with the middle term at A for $\ell_1 = 0$ and $\ell_2 = \ell$.

Then $2M_1\ell + M_2\ell = -\frac{p\ell^3}{4} = 3M_1\ell$, or $M_1 = -\frac{p\ell^2}{12}$

Also $M_c = \frac{p\ell^2}{8} - \frac{p\ell^2}{12} = +\frac{p\ell^2}{24}$.

Also $M_m = \frac{Pab}{\ell} - \frac{bM_1}{\ell} - \frac{aM_2}{\ell} = \frac{Pab}{\ell}\left[1 - \frac{b^3}{\ell^2} - \frac{a^3}{\ell^2}\right] = \frac{Pab}{\ell^3}(\ell^2-a^2-b^2) = \frac{2Pa^2b^2}{\ell^3}$.

Problem 28D. A closed 4 sided frame with uniform loading on all sides; Solution by the Three Moment Equation $M_1\frac{l_1}{I_1} + 2M_2\left(\frac{l_1}{I_1}+\frac{l_2}{I_2}\right) + M_3\frac{l_2}{I_2} = -\Sigma\frac{wl^2}{4}\frac{l}{I}$, - - - - - - - - 28B.



Given the unit loads p, w, and e, acting on the 4 sides of the frame, find the moments M_1 and M_2 at the corners.

Let $\frac{pl^2}{8} = P$, $\frac{eh^2}{8} = E$ and $\frac{wl^2}{8} = W$.

Now apply the 3-moment equation, first with the middle term at b and again with the middle term at a and obtain:

$$M_2\frac{l}{I_2} + 2M_2\left(\frac{l}{I_2}+\frac{h}{I_3}\right) + M_1\frac{h}{I_3} = -\frac{wl^2}{4}\cdot\frac{l}{I_2} - \frac{eh^2}{4}\cdot\frac{h}{I_3} = -2W\frac{l}{I_2} - 2E\frac{h}{I_3},$$

and

$$M_2\frac{h}{I_3} + 2M_1\left(\frac{h}{I_3}+\frac{l}{I_1}\right) + M_1\frac{l}{I_1} = -\frac{eh^2}{4}\cdot\frac{h}{I_3} - \frac{pl^2}{4}\cdot\frac{l}{I_1} = -2P\frac{l}{I_1} - 2E\frac{h}{I_3}$$

which equations simplify into the following and are easily solved for M_1 and M_2 when numerical values are inserted.

$$\left.\begin{aligned}\left(\frac{3l}{I_2}+\frac{2h}{I_3}\right)M_2 + \frac{h}{I_3}M_1 &= -2W\frac{l}{I_2} - 2E\frac{h}{I_3}\\ \frac{h}{I_3}M_2 + \left(\frac{3l}{I_1}+\frac{2h}{I_3}\right)M_1 &= -2P\frac{l}{I_1} - 2E\frac{h}{I_3}\end{aligned}\right\}$$

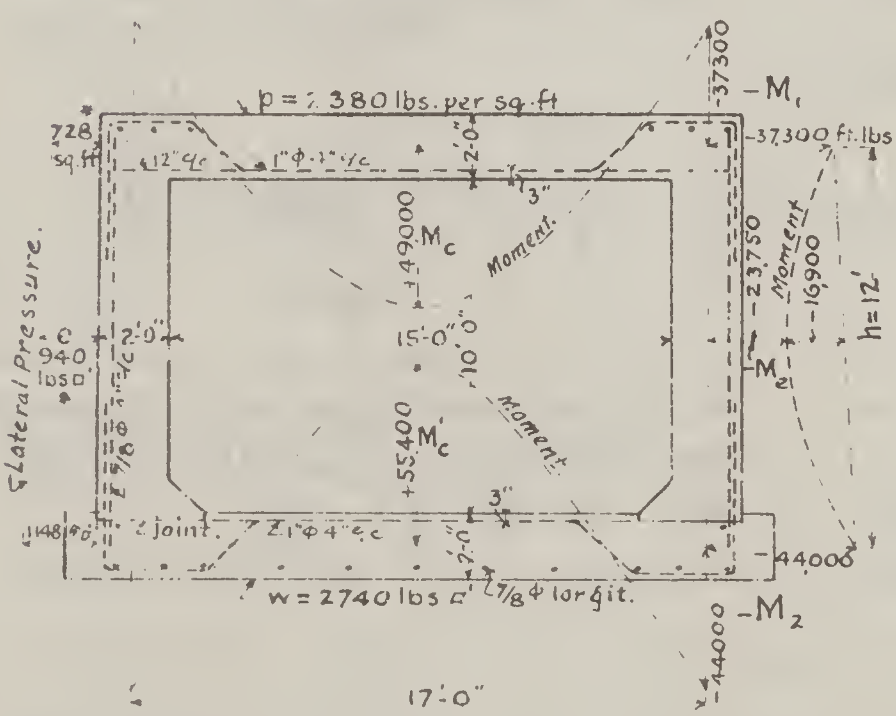
Moments of inertia in these equations may be in any units, since each term involves a factor $\frac{1}{I}$. ...... 28E.

When $I_1 = I_2 = I_3$ or I constant, then the above equations give

$$M_1 = \frac{(eh^3+pl^3)(3l+2h) - h(wl^3+eh^3)}{4h^2 - 4(3l+2h)^2} \quad \text{and} \quad M_2 = -\frac{(3l+2h)}{h}M_1 - \frac{eh^3+pl^3}{4h} \text{. - - - 28F.}$$

When $p = w$, then $M_1 = M_2 = -\frac{eh^3+pl^3}{12(l+h)}$. Having found M_1 and M_2, then the other moments are: $M_c = \frac{pl^2}{8} - M_1$; $M'_c = \frac{wl^2}{8} - M_2$; $M_e = \frac{eh^2}{8} - \frac{M_1+M_2}{2}$

Numerical Example.*



We will now solve the case here shown.

$\frac{pl^2}{8} = 86300$ ft.lbs. $\frac{eh^2}{8} = 16900$ ft.lbs.

$\frac{wl^2}{8} = 99400$ ft.lbs. I constant.

$pl^3 = 11,680,000$; $eh^3 = 1,624,000$; $wl^3 = 13,450,000$.

$M_1 = \frac{13,304,000 \times 75 - 15,074,000 \times 12}{4 \times 144 - 4 \times 75 \times 75} = -37,300$ ft.lbs

$M_2 = -\frac{75}{12}M_1 - \frac{13,304,000}{4 \times 12} = -44,000$ ft.lbs.

$M_c = 86,300 - 37,300 = +49,000$ ft.lbs.

$M'_c = 99,400 - 44,000 = +55,400$ ft.lbs

$M_e = 16,900 - \frac{37300+44000}{2} = -23,750$ ft.lbs.

The reinforcing steel for these moments is calculated for $f_s = 16000^{\#}\square''$, $f_c = 650^{\#}\square''$ using the formula for steel area $A_s = \frac{M}{1165d}$, where M is in ft.lbs, and effect. depth $d = 21''$ for all sides and 29" at corners.

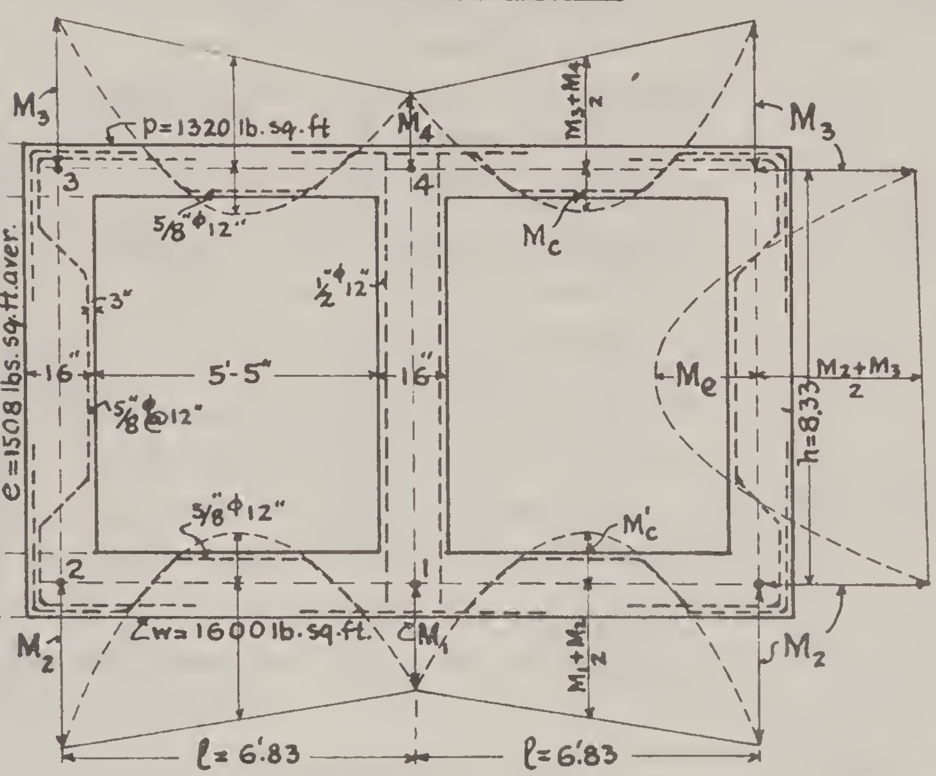
The moment curves can be drawn by plotting the 8 moments as shown and the points of contraflexure are thus located. Note that all bent bars are alike, a condition often attainable and desirable.

*The above example is a culvert under a railway embankment with 12 ft. of earth fill over the top with Cooper's E50 loading on the track. The lateral earth pressure $e = wH\tan^2(45 - \frac{\varphi}{2})$ lbs. p. sq. ft. This gives $e = 35H$ for $\varphi = 30°$, $w = 105$ lbs. p. cu. ft. and H = height of fill in feet, including surcharge for Live Load.

David A. Molitor, 9/2 '26.

Problem 28E. Double Culvert, Uniform Loading, by Three-Moment Equation.

14 ft Fill at 80 lbs. p. cu. ft.



Unit stresses $f_s = 16000^{\#}$, $f_c = 650^{\#}\square''$

$A_s = \frac{M}{1165d}$, I is taken constant.

Load data.

$p = 1320^{\#}\square'$, $\frac{pl^2}{8} = 7{,}700$ ft.lbs. $\frac{pl^3}{4} = 105{,}000$.

$w = 1600^{\#}\square'$, $\frac{wl^2}{8} = 9{,}340$ ft.lbs. $\frac{wl^3}{4} = 127{,}200$.

$e = 1508^{\#}\square'$, $\frac{eh^2}{8} = 13{,}100$ ft.lbs. $\frac{eh^3}{4} = 218{,}000$.

The problem involves 6 unknown moments of which two pairs are equal by symmetry of loading and dimensions.

Hence we solve for 4 moments by writing 4 equations with unknowns M_1, M_2, M_3 and M_4. This is done by applying the 3-moment equation four times, with the middle term successively at 1, 2, 3 and 4.

General Eq. 28B.- $M_1 l_1 + 2M_2(l_1 + l_2) + M_3 l_2 = -\Sigma \frac{wl^3}{4}$ - - - (28B)

Numerical Equations by substitution of values from the Figure.

$6.83 M_2 + 27.32 M_1 + 6.83 M_2 = -254{,}400$ for middle term at 1.
$6.83 M_1 + 30.3 M_2 + 8.33 M_3 = -345{,}200$ " " " " 2.
$8.33 M_2 + 30.3 M_3 + 6.83 M_4 = -323{,}000$ " " " " 3.
$6.83 M_3 + 27.32 M_4 + 6.83 M_3 = -210{,}000$ " " " " 4.

Collect terms in (1) and (4), divide each equation by the coefficient of its first term, and obtain the simplified derivatives (b).

$$\left.\begin{array}{llr}
M_1 + 0.5 M_2 & = -9{,}320 & (1)\\
M_1 + 4.44 M_2 + 1.22 M_3 & = -50{,}600 & (2)\\
M_2 + 3.64 M_3 + 0.82 M_4 & = -38{,}800 & (3)\\
M_3 + 2.00 M_4 & = -15{,}350 & (4)
\end{array}\right\} b$$

(2) minus (1) $3.94 M_2 + 1.22 M_3 = -41{,}280$

or $M_2 + 0.31 M_3 = -10{,}450$ (5)

(3) minus (5) $3.33 M_3 + 0.82 M_4 = -28{,}350$

or $M_3 + 0.246 M_4 = -8{,}500$ (6)

(4) minus (6) $1.754 M_4 = -6{,}850$

$M_4 = -3{,}900$ ft.lbs.

From (6) $M_3 = -7{,}600$ " "

From (5) $M_2 = -8{,}094$ " "

From (1) $M_1 = -5{,}250$ " "

The center Moments.

Having found the 4 moments M_1, M_2, M_3 and M_4, we now find the center moment of each side as follows:

$M_c = \frac{pl^2}{8} - \frac{M_3+M_4}{2} = +1950$ ft.lbs.

$M'_c = \frac{wl^2}{8} - \frac{M_1+M_2}{2} = +2668$ " "

$M_e = \frac{eh^2}{8} - \frac{M_2+M_3}{2} = +5253$ " "

The required steel per foot of culvert is now figured for the respective moments at corners and centers of all sides, from $A_s = \frac{M}{1165d}$, for M in ft.lbs, and d = net thickness of slab in inches.

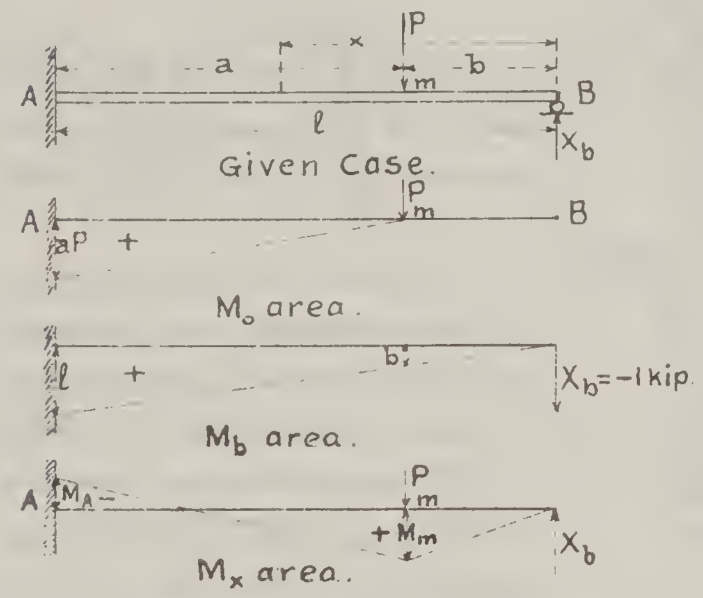
The middle wall has no stress except as a column, but a nominal reinforcement is provided to take care of temperature and a possible one sided water pressure.

David A. Molitor 9/2/26

Art. 29. Statically Indeterminate Beams and Rigid Frames.

Problem 29A. Beam fixed at one end and simply supported at the other end. Single load P.

There is one redundant condition, see Problem 6A. In the present analysis the end reaction is treated as the redundant and the principal system is thus a cantilever beam fixed at A.



The general moment equation for the moment M_x at any point distant x from B, is by Eq 19B,

$$M_x = M_o - M_b X_b = P(x-b) - xX_b \quad \text{- - - - - - - -} \quad (1)$$

The elasticity equation for one redundant X_b is by Eq.(19C), when the support at B is rigid.

$$\frac{1}{EI}\int M_o M_b\,dx = \frac{X_b}{EI}\int M_b^2\,dx \quad \text{- - - - - - - -} \quad (2)$$

First draw the M_o and the M_b moment diagrams for the load P and the conventional load $X_b=-1$, observing that both are positive for the present assumptions indicated on the respective diagrams.

The two integrals in Eq(2) are now evaluated using the substitution formulas of case 3, Table 26A.

$\int_m^\ell M_o M_b\,dx = \frac{a^2P}{6}(2\ell+b)$ for the length a common to both the M_o and the M_b areas.

and $\int_0^\ell M_b^2\,dx = \frac{\ell}{3}\times\ell^2 = \frac{\ell^3}{3}$ for the length ℓ.

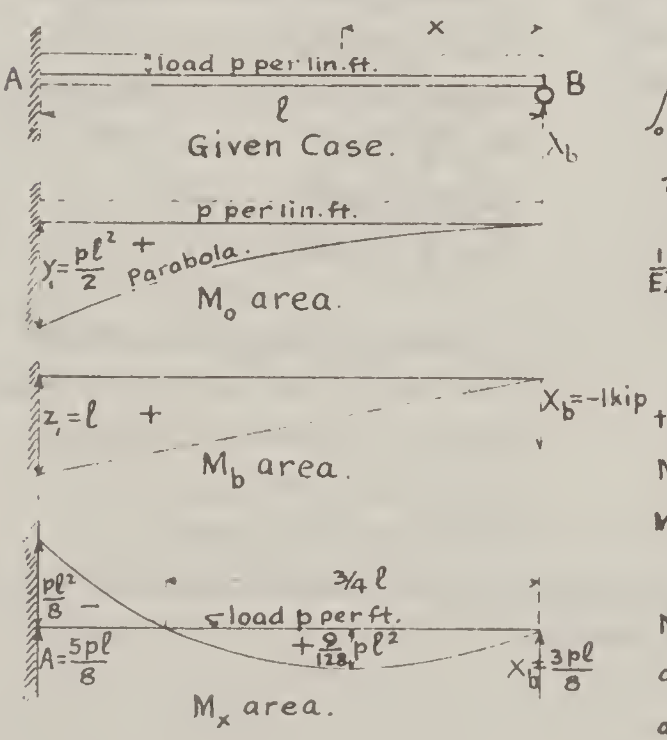
These values substituted into the above Eq.(2), furnish the redundant X_b as follows:

$\frac{1}{EI}\times\frac{a^2P}{6}(2\ell+b) = \frac{X_b}{EI}\cdot\frac{\ell^3}{3}$, and since EI is constant, then $X_b=\frac{a^2P}{2\ell^3}(2\ell+b)=R_B$, which inserted in Eq(1) gives---$M_x = P(x-b)-\frac{a^2Px}{2\ell^3}(2\ell+b)$, as the moment equation for any point x between A and m.

When $x=b$, then $M_m = -\frac{a^2bP}{2\ell^3}(2\ell+b)$; and when $x=\ell$, then $M_A = P(\ell-b)-\frac{a^2\ell P}{2\ell^3}(2\ell+b)=\frac{Pab}{2\ell^2}(\ell+b)$.

We can now draw the complete M_x diagram as shown, noting that the actual moments are of opposite signs.

Problem 29B. Beam fixed at one end and simply supported at the other. Uniform Load p per ft.



This problem is exactly like 29A except the M_o area.

The two integrals in Eq. 2 are evaluated as per Cases 2 & 8.

$\int_0^\ell M_o M_b\,dx = \frac{\ell}{4}\times\frac{p\ell}{2}\times\ell = \frac{p\ell^4}{8}$, from Case 8, Table 26A.

$\int_0^\ell M_b^2\,dx = \frac{\ell}{3}\times\ell^2=\frac{\ell^3}{3}$, from Case 2, Table 26A.

The above Eq.(2) again gives the redundant X_b, as

$\frac{1}{EI}\frac{p\ell^4}{8}=\frac{1}{EI}\times\frac{\ell^3}{3}X_b$ or $X_b=\frac{3p\ell}{8}$ = reaction at B.

The reaction at $A = p\ell - X_b = \frac{5p\ell}{8}$.

Substituting the value $X_b=\frac{3p\ell}{8}$ into Eq.(1) gives the moment M_x at any point distant x from B, as

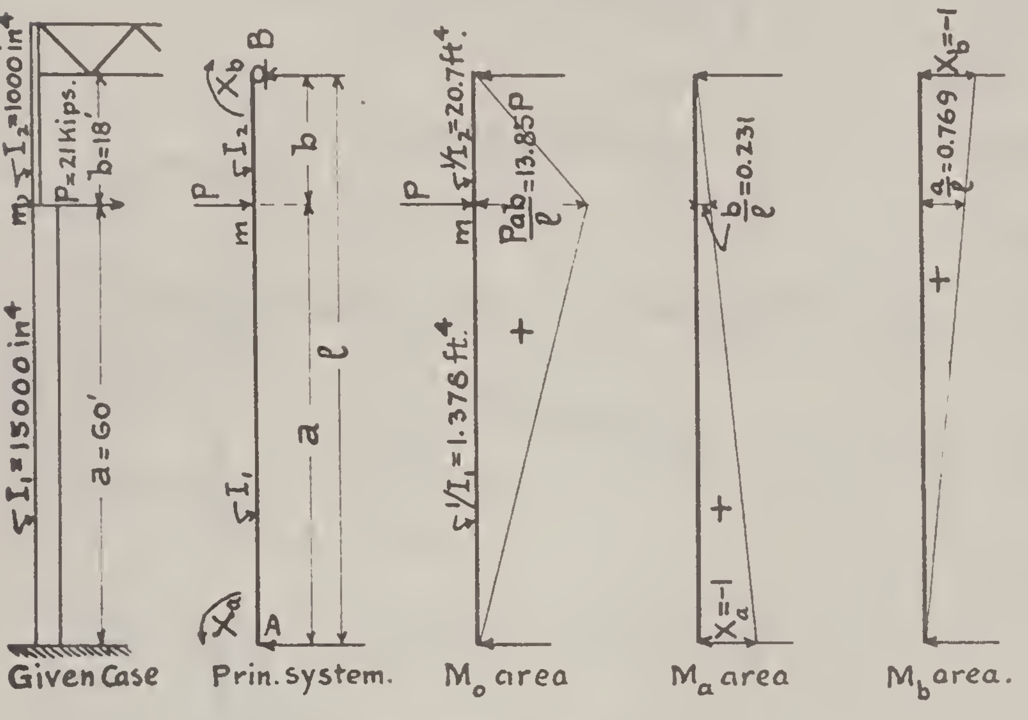
$$M_x = M_o - M_b X_b = \frac{px^2}{2} - xX_b = \frac{px^2}{2}-\frac{3p\ell x}{8}=\frac{px}{2}\left(x-\frac{3\ell}{4}\right) \text{.....} (3)$$

$M_x=0$ when $x=\frac{3\ell}{4}$ and when $x=\ell$, $M_x=M_A=\frac{p\ell^2}{2}-\frac{3p\ell^2}{8}=\frac{p\ell^2}{8}$.

The equation $M_x=\frac{px}{2}\left(x-\frac{3\ell}{4}\right)$ enables us to draw the entire M_x moment diagram for the actual Loading shown and it should be noted as in Problem 29A, that the actual signs of the moments are opposite to those found from Eq.(1) because the abscissa x is really taken in a negative direction.

Problem 29C. Column fixed top and bottom, variable Moment of Inertia, hor. Load P.

Simple beam as principal system with end moments X_a and X_b as redundants.
Required to find the end moments and M_m at m, also the horizontal deflection δ_m at m.



Data.

$I_1=15000\,in^4=0.725\,ft^4.\quad \frac{1}{I_1}=1.378$

$I_2=1000\,in^4=0.0483\,ft^4.\quad \frac{1}{I_2}=20.7$

$P=21$ kips, $a=60'$, $b=18'$, $\ell=78'$.

This problem involves only two redundants, because the column is free to expand vertically.

The Moment Diagrams

for M_o, M_a and M_b here shown, furnish the data for evaluating the two redundant moments from the elasticity equations 19C and M_m is then found from Eq. 19B.

The elasticity equations for unyielding supports are:

$$\left.\begin{aligned}\frac{X_a}{I}\int M_a^2\,dx+\frac{X_b}{I}\int M_aM_b\,dx&=\frac{1}{I}\int M_oM_a\,dx\\ \frac{X_a}{I}\int M_bM_a\,dx+\frac{X_b}{I}\int M_b^2\,dx&=\frac{1}{I}\int M_oM_b\,dx\end{aligned}\right\}(1)$$

The evaluation of the integrals is from Cases 2 & 3 T.26A.

$$\frac{1}{I}\int M_oM_a\,dx=\frac{1.378a}{6}\times\frac{Pab}{\ell}\left(\frac{2b}{\ell}+1\right)+\frac{20.7b}{6}\times\frac{Pab}{\ell}\left(\frac{2b}{\ell}\right)=676P.$$

$$\frac{1}{I}\int M_oM_b\,dx=\frac{1.378a}{6}\times\frac{Pab}{\ell}\times\frac{2a}{\ell}+\frac{20.7b}{6}\times\frac{Pab}{\ell}\left[\frac{2a}{\ell}+1\right]=2478P.$$

$$\frac{1}{I}\int M_aM_b\,dx=\frac{1.378a}{6}\times\frac{a}{\ell}\left[\frac{2b}{\ell}+1\right]+\frac{20.7b}{6}\times\frac{b}{\ell}\left[\frac{2a}{\ell}+1\right]=51.9$$

$$\frac{1}{I}\int M_a^2\,dx=\frac{1.378a}{3}\left[1+\frac{b}{\ell}+\frac{b^2}{\ell^2}\right]+\frac{20.7b}{3}\times\frac{b^2}{\ell^2}=42.0$$

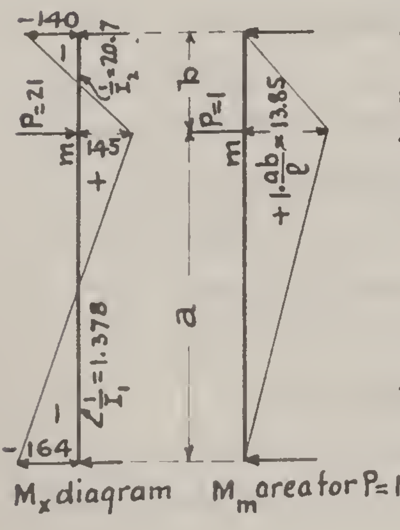
$$\frac{1}{I}\int M_b^2\,dx=\frac{1.378a}{3}\times\frac{a^2}{\ell^2}+\frac{20.7b}{3}\left[\frac{a^2}{\ell^2}+\frac{a}{\ell}+1\right]=310.7$$

Introducing the numerical integrals into Eqs.(1)

then
$$42.0X_a+51.9X_b=676P$$
$$51.9X_a+310.7X_b=2478P$$
or
$$X_a+1.234X_b=16.05P$$
$$X_a+5.980X_b=47.70P$$
$$4.746X_b=31.65P$$

Finally, $X_b=6.675P=140.0$ kip ft., and $X_a=7.805P=164.0$ kip ft. which values into Eq.19B give, $M_m=M_o-M_aX_a-M_bX_b=\frac{Pab}{\ell}-\frac{b}{\ell}X_a-\frac{a}{\ell}X_b=21\times13.85-0.231\times164-0.769\times140=145.3$ kip ft.

Horizontal Deflection δ_m at point m. With X_a, X_b and M_m given, draw the M_x diagram as shown, also draw the M_m diagram for P=1 acting at m. Then $\delta_m=\frac{1}{EI}\int M_xM_m\,dx$. From Table 26A, case 3, evaluate the integral over the lengths a and b, noting that the end ordinates of the M_x diagram are negative moments.

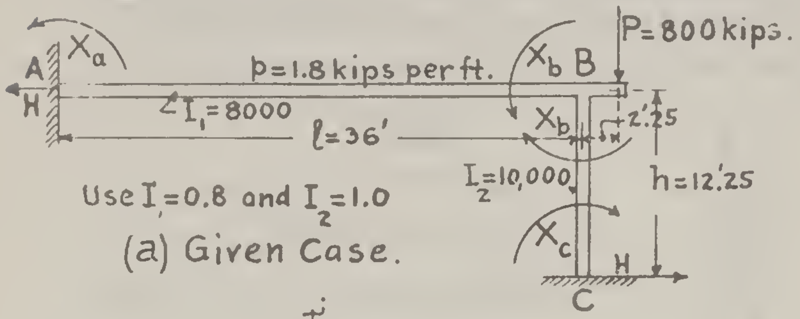


Then $E\delta_m=1.378\times\frac{ab}{\ell}\times\frac{a}{6}\left[2\times145.3-164\right]+20.7\times\frac{ab}{\ell}\times\frac{b}{6}\left[2\times145.3-140\right]=148{,}766$.

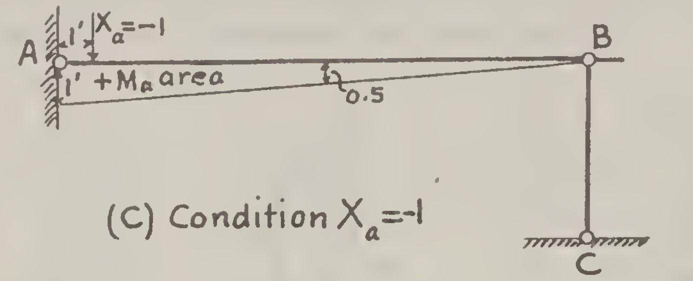
and $\delta_m=\frac{148766}{E}=\frac{148766}{4{,}320{,}000}=0.0344$ ft. $=0.413$ in.

This same problem should be solved by area moments, Art.22, observing that the end reactions of the elastic weights must be zero in the present problem, because the end tangents to the elastic curve are zero, giving $\tan\beta_1=\frac{R_1}{EI}=0$, or $R_1=0$, by Art. 24. This condition always obtains at the fixed end of any restrained beam.

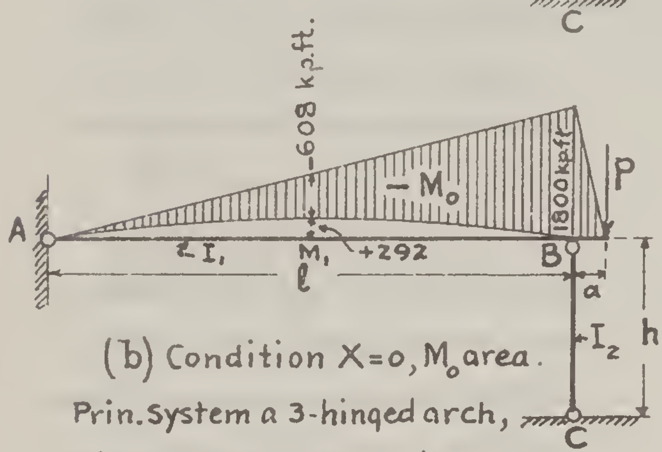
Problem 29D. Column and Cantilever beam supporting loads as shown. Find Moments at A, B and C, also the horizontal thrust at A and C. Connections at A, B and C are rigid.



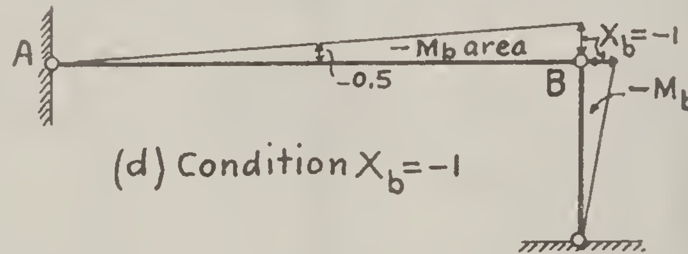
(a) Given Case.



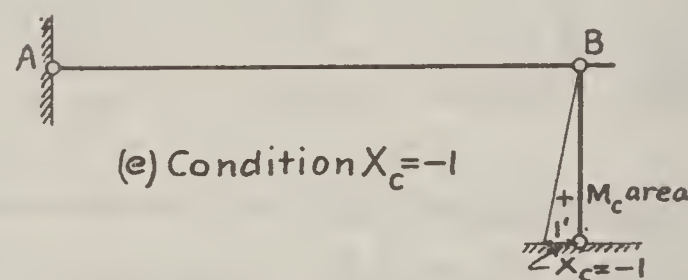
(C) Condition X_a=-1



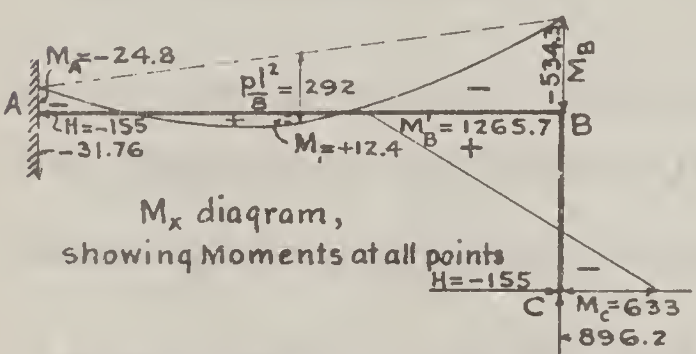
(b) Condition X=o, M_o area.
Prin. System a 3-hinged arch, with moment aP applied at B.



(d) Condition X_b=-1



(e) Condition X_c=-1



M_x diagram, showing Moments at all points

Elasticity Equations, unyielding supports.

$$\frac{X_a}{I}\int M_a^2dx + \frac{X_b}{I}\int M_aM_bdx + \frac{X_c}{I}\int M_aM_cdx = \frac{1}{I}\int M_oM_adx$$

$$\frac{X_a}{I}\int M_bM_adx + \frac{X_b}{I}\int M_b^2dx + \frac{X_c}{I}\int M_bM_cdx = \frac{1}{I}\int M_oM_bdx$$

$$\frac{X_a}{I}\int M_cM_adx + \frac{X_b}{I}\int M_cM_bdx + \frac{X_c}{I}\int M_c^2dx = \frac{1}{I}\int M_oM_cdx$$

Moment Equations according to Eq.19B.

$M_x = M_o - M_aX_a - M_bX_b - M_cX_c$ for any point x.

$M_A = 0 - 1.X_a - 0 - 0 = -24.8$ kip ft.

$M_B = -aP - 0 + 1.X_b - 0 = -534.3$ on the beam.

$M'_B = 0 - 0 + 1.X_b - 0 = +1265.7$ top of Col.

$M_c = 0 - 0 - 0 - 1.X_c = -633.0$ at base of Col.

$M_1 = -608 - 0.5X_a + 0.5X_b - 0 = +12.4$ at cent. of beam.

$H = H_o - H_aX_a - H_bX_b - H_cX_c = 0 - 0 - \frac{X_b}{h} - \frac{X_c}{h} = -155$ kips.

Note that $M_B + M'_B = 1800$ kip ft = the applied ext. mom.

To evaluate the above integrals we employ Table 26A, Case 14, for the 3 integrals involving M_o and first find m_1 at A equal to 651460, and m_2 at B equal to 262660, as the static moments of the M_o area about A and B. The other integrals are found by applying Case 2. - The following numerical values are thus obtained:

$\frac{1}{I}\int M_oM_adx = -\frac{1}{I_1\ell}\left[1\times262660 + 0\right] = -\frac{262660}{0.8\times36} = -9,120$

$\frac{1}{I}\int M_oM_bdx = +\frac{1}{I_1\ell}\left[0 + 1\times651460\right] = +\frac{651460}{0.8\times36} = +22,620$

$\frac{1}{I}\int M_oM_cdx = 0$

$\frac{1}{I}\int M_aM_bdx = -\frac{\ell}{6I_1}\left[1\times1 + 0\right] = -\frac{36}{6\times0.8} = -7.5$

$\frac{1}{I}\int M_aM_cdx = 0$

$\frac{1}{I}\int M_bM_cdx = -\frac{h}{6I_2}\left[1\times1 + 0\right] = -\frac{12.25}{6\times1.0} = -2.042$

$\frac{1}{I}\int M_a^2dx = \frac{\ell}{3I_1}\times1 = \frac{36}{3\times0.8} = +15.0$

$\frac{1}{I}\int M_b^2dx = \frac{\ell}{3I_1}\times1 + \frac{h}{3I_2}\times1 = \frac{36}{3\times0.8} + \frac{12.25}{3\times1} = +19.083$

$\frac{1}{I}\int M_c^2dx = \frac{h}{3I_2} = \frac{12.25}{3\times1} = +4.083$

The 3 X's came out + showing assumptions (a) were correct.

Numerical Equations.

$15X_a - 7.5X_b + 0 = -9,120$

$-7.5X_a + 19.083X_b - 2.042X_c = +22620$

$0 - 2.042X_b + 4.083X_c = 0$

$X_a - 0.5X_b \;----\; = -608$

$-X_a + 2.542X_b - 0.272X_c = +3,016$

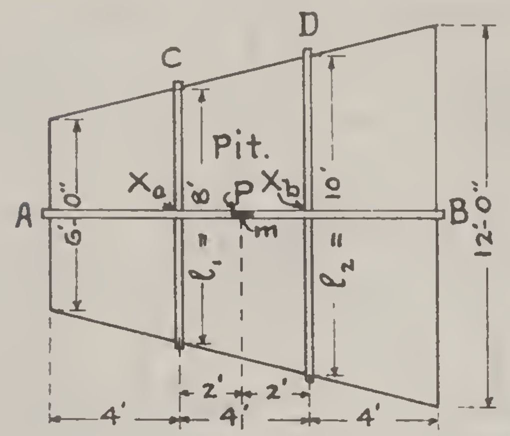
$2.042X_b - 0.272X_c = 2,408$

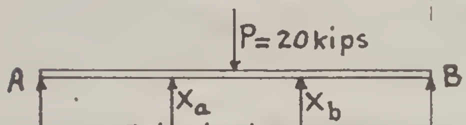
$-2.042X_b + 4.083X_c = 0$

$3.811X_c = 2408$

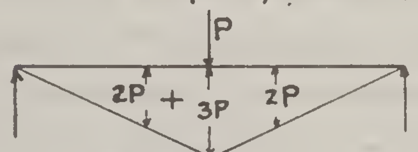
$X_c = +633.0$ kip ft., $X_b = +1265.7$ and $X_a = +24.8$ kip ft.

Problem 29 E. Column supported on Three Beams over a Pit as shown in figure. Find Bending Moments in the Beams and the deflection under the Column.

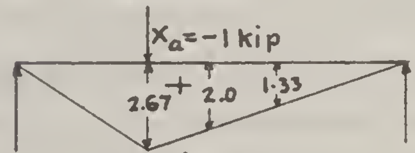




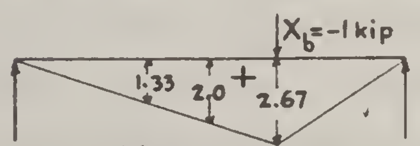
Principal system.



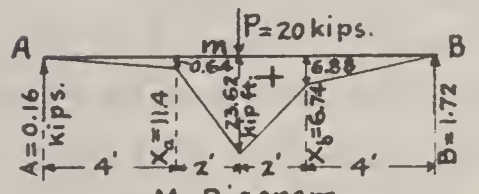
M_o Diagram



M_a Diagram



M_b Diagram



M_x Diagram.

The 3 beams are 10" I 22.4#, I = 113.6 in.^{4} = 0.00548 ft.^{4}, S = 22.72 in.^{3}

The column with load P = 20 kips, is supported on the beam AB, and the beam AB is supported on the pit walls at A and B, and on two cross-beams C and D, which latter afford elastic supports X_a, and X_b, and these are treated as redundants. The principal system is the beam A-B, simply supported at A and B.

The elasticity Equations 19C for 2 redundants, are:

$$\left.\begin{aligned}\delta_a &= \frac{1}{EI}\int M_o M_a dx - \frac{X_a}{EI}\int M_a^2 dx - \frac{X_b}{EI}\int M_a M_b dx = \frac{10.67 X_a}{EI}\\ \delta_b &= \frac{1}{EI}\int M_o M_b dx - \frac{X_a}{EI}\int M_b M_a dx - \frac{X_b}{EI}\int M_b^2 dx = \frac{20.83 X_b}{EI}\end{aligned}\right\} \text{-----(1)}$$

In this problem δ_a and δ_b are not zero, and must be evaluated in terms of the unknown loads X_a and X_b, as the deflections at the centers of the cross-beams as follows:

$$\delta_a = \frac{X_a \ell_1^3}{48EI_1} = \frac{10.67X_a}{EI_1}; \text{ and } \delta_b = \frac{X_b \ell_2^3}{48EI_2} = \frac{20.83\,X_b}{EI_2}.$$

Since EI occurs in each term of the two Eqs (1) and is constant, we can cancel it from all terms.

The integrals are now evaluated from Cases 3 and 4, Table 26A.

$$\int M_o M_a dx = \frac{4\times 2P}{6}(2\times 2.67) + \frac{2}{6}\left[2P(5.34+2) + 3P(2.67+4)\right] + \frac{6\times 3P}{6}\times 4 = 30.68P \text{---(1)}$$

$$\int M_o M_b dx = \frac{6\times 3P}{6}\times 4 + \frac{2}{6}\left[3P(4+2.67) + 2P(2+5.34)\right] + \frac{4\times 2P}{6}(2+2.67) = 30.68P \text{--(2)}$$

$$\int M_a M_b dx = \frac{4\times 2.67}{6}\times 2.67 + \frac{4}{6}\left[2.67(2.67+2.67) + 1.33(1.33+5.34)\right] + \frac{4\times 1.33}{6}\times 5.34 = 24.88 \text{--(3)}$$

$$\int M_a^2 dx = \frac{4}{3}\times \overline{2.67}^2 + \frac{8}{3}\times \overline{2.67}^2 = 28.4 \text{ ---------------------(4)}$$

$$\int M_b^2 dx = \frac{8}{3}\times \overline{2.67}^2 + \frac{4}{3}\times \overline{2.67}^2 = 28.4 \text{ --------------------- (5)}$$

Note here the equality between (1) and (2) and between (4) and (5).

The numerical equations and their solution follow:

$$\left.\begin{aligned}30.68P - 28.4X_a - 24.88X_b &= 10.67X_a\\ 30.68P - 24.88X_a - 28.4X_b &= 20.83X_b\end{aligned}\right\} \text{------------- (6)}$$

$$39.07X_a + 24.88X_b = 30.68P$$

$$24.88X_a + 49.23X_b = 30.68P$$

$$X_a + 0.637X_b = 0.785P$$

$$X_a + 1.975X_b = 1.235P$$

$$1.338X_b = 0.450P \text{ or } X_b = 0.337P \text{ and } X_a = 0.57P.$$

According to Eq. 19B, the M_x equation is-

$M_x = M_o - M_a X_a - M_b X_b = M_o - 0.57PM_a - 0.337PM_b$.

Hence $M_m = 3P - 2\times 0.57P - 2\times 0.337P = 1.186P = 23.62$ kip ft.

and stress in beam AB is $f = \frac{M_m}{S} = \frac{23620\times 12}{22.72} = 12{,}500$ lbs.p.sq.in.

Reaction $A = A_o - A_a X_a - A_b X_b = 10 - \frac{2}{3}X_a - \frac{1}{3}X_b = 0.16$ kip; Reaction $B = B_o - B_a X_a - B_b X_b = 10 - \frac{1}{3}X_a - \frac{2}{3}X_b = 1.72$ kips.

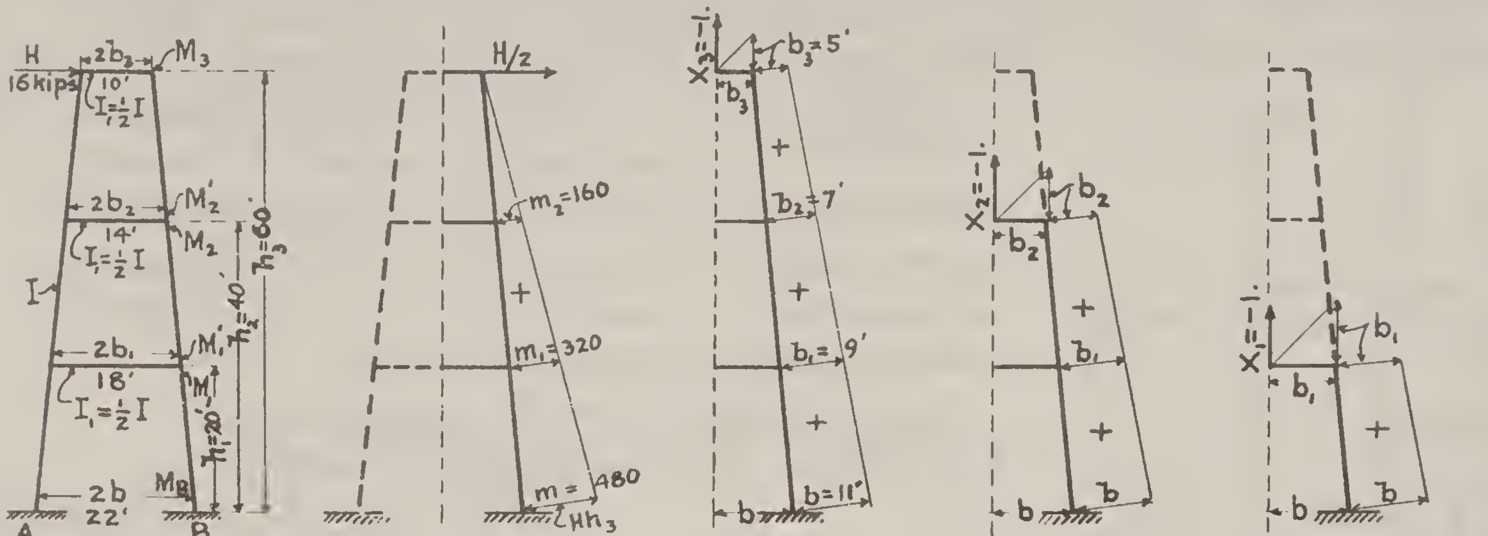
and $A + B + X_a + X_b = 20.02$ kips $= P$. The reactions $C = \frac{1}{2}X_a = 5.7$ kips, and reactions $D = \frac{1}{2}X_b = 3.37$ kips.

The Deflection $\delta_m = \frac{1}{EI}\int M_x M_{o-1} dx$ and is obtained by integrating the M_x area against the M_o area for P=1.

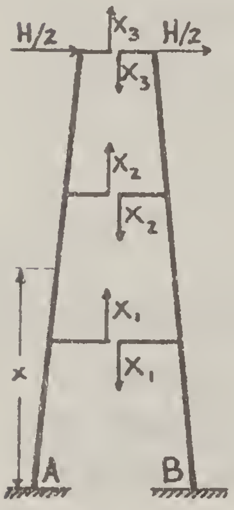
Thus $\int M_x M_{o-1} dx = \frac{4}{6}\times 0.64\times 4 + \frac{2}{6}\left[0.64(4+3) + 23.62(2+6)\right] + \frac{2}{6}\left[23.62(6+2) + 6.88(3+4)\right] + \frac{4}{6}\times 6.88\times 4 = 163.57$

For E = 4,320,000 kips per sq.ft. and I = 0.00548 ft.^{4}, $\delta_m = \frac{163.57}{4{,}320{,}000\times 0.00548} = 0.00691$ ft. $= 0.083$ inch.

Problem 29F. Three Story Bent, Rigid Reinforced Concrete Frame.



Given Case. Condition X=0. M_0 Area. Condition $X_3=-1$. M_3 Area. Condition $X_2=-1$. M_2 Area. Condition $X_1=-1$. M_1 Area.



Principal System, 2-Cantilever Posts.

We cut the bent into two equal halves, obtaining 2 cantilever posts for the principal system. The right half is analysed for Moments M_B, M_1, M_2, M_3, etc., noting that the moments for the left leg will be equal and of opposite sign to those found.

The problem actually involves 9 redundants – a shear, a moment and a thrust at each of the beams cut. The moments are zero because for equal end moments on the beams, the points of contraflexure are at the centers where the section is taken. The thrusts are zero except at beams where horizontal loads are applied, as at the top, and then we may assume that a load as H is divided equally on the two cantilevers. Hence only three shears X_1, X_2 and X_3 must be evaluated as redundants.

The vertical displacements δ_1, δ_2 and δ_3 of the points of application of the redundants in the direction of the X's are also zero, hence the elasticity Eqs. 19C, become zero equations as follows:

$$\frac{X_1}{I}\int M_1^2dx + \frac{X_2}{I}\int M_1M_2dx + \frac{X_3}{I}\int M_1M_3dx = \frac{1}{I}\int M_0M_1dx$$

$$\frac{X_1}{I}\int M_2M_1dx + \frac{X_2}{I}\int M_2^2dx + \frac{X_3}{I}\int M_2M_3dx = \frac{1}{I}\int M_0M_2dx$$

$$\frac{X_1}{I}\int M_3M_1dx + \frac{X_2}{I}\int M_3M_2dx + \frac{X_3}{I}\int M_3^2dx = \frac{1}{I}\int M_0M_3dx$$

The integrals have the following values:

$$\frac{1}{I}\int M_0M_1dx = \frac{h_1}{6I}\left[m_1(2b_1+b)+m(b_1+2b)\right] = +80{,}533$$
$$\frac{1}{I}\int M_0M_2dx = \frac{h_2}{6I}\left[m_2(2b_2+b)+m(b_2+2b)\right] = +119{,}470$$
$$\frac{1}{I}\int M_0M_3dx = \frac{h_3}{6I}\left[m_3(2b_3+b)+m(b_3+2b)\right] = +129{,}600$$
$$\frac{1}{I}\int M_1M_2dx = \frac{1}{I}\int M_2M_1dx = \frac{h_1}{3I}\left[b_1^2+b_1b+b^2\right] = +2{,}007$$
$$\frac{1}{I}\int M_1M_3dx = \frac{1}{I}\int M_3M_1dx = \frac{h_1}{3I}\left[b_1^2+b_1b+b^2\right] = +2{,}007$$
$$\frac{1}{I}\int M_2M_3dx = \frac{1}{I}\int M_3M_2dx = \frac{h_2}{3I}\left[b_2^2+b_2b+b^2\right] = +3{,}293$$
$$\frac{1}{I}\int M_1^2dx = \frac{h_1}{3I}\left[b_1^2+b_1b+b^2\right] + \frac{b_1}{3I_1}\left[0+0+b_1^2\right] = +2{,}493$$
$$\frac{1}{I}\int M_2^2dx = \frac{h_2}{3I}\left[b_2^2+b_2b+b^2\right] + \frac{b_2^3}{3I_1} \text{ - - - - - - - } = +3{,}521$$
$$\frac{1}{I}\int M_3^2dx = \frac{h_3}{3I}\left[b_3^2+b_3b+b^2\right] + \frac{b_3^3}{3I_1} \text{ - - - - - } = +4{,}103$$

Evaluation of the Integrals.

The integrals can all be evaluated as per Case 4, Table 26A, for two trapazoids Z and Y as

$$\int M_zM_ydx = \frac{\ell}{6}\left[z_1(2y_1+y_2)+z_2(y_1+2y_2)\right]$$

and

$$\int M_z^2dx = \frac{\ell}{3}\left[z_1^2+z_1z_2+z_2^2\right].$$

Numerical Equations.

$$2{,}493X_1 + 2{,}007X_2 + 2{,}007X_3 = 80{,}533$$
$$2{,}007X_1 + 3{,}521X_2 + 3{,}293X_3 = 119{,}470$$
$$2{,}007X_1 + 3{,}293X_2 + 4{,}103X_3 = 129{,}600$$

$$X_1 + 0.806X_2 + 0.806X_3 = 32.3$$
$$X_1 + 1.752X_2 + 1.638X_3 = 59.5$$
$$X_1 + 1.638X_2 + 2.042X_3 = 64.5$$

$$0.946X_2 + 0.832X_3 = 27.2$$
$$-0.114X_2 + 0.404X_3 = 5.0$$

$$X_2 + 0.88X_3 = 28.7$$
$$-X_2 + 3.54X_3 = 43.8$$

$X_1 = +7.64$. $X_2 = +14.28$ $4.42X_3 = 72.5$; $X_3 = 16.4$

Problem 29F, Continued.

Having found the three redundant Shears $X_1 = +7.64$ kips, $X_2 = 14.28$ kips, and $X_3 = 16.4$ kips, we now employ Eq. 19B to find the moments M_x for various points of the principal system for certain heights x above the ground. Then for the right hand post we find:

when $x = h_3$ at top of post $\quad M_3 = 0 - b_3X_3 \text{ ------------------- } = -\ 82.0$ kipft.

" $x = h_2$ just above joint $\quad M_2' = m_2 - b_2X_3 \text{ ------- } = \frac{H}{2}(h_3 - h_2) - b_2X_3 \text{ ---- } = +\ 45.2$ " "

" " just below joint $\quad M_2 = m_2 - b_2X_2 - b_2X_3 \text{ ---- } = \frac{H}{2}(h_3 - h_2) - b_2(X_2 + X_3) \text{ -- } = -\ 54.7$ " "

" $x = h_1$ just above joint $\quad M_1' = m_1 - b_1X_2 - b_1X_3 \text{ --- } = \frac{H}{2}(h_3 - h_1) - b_1(X_2 + X_3) \text{ -- } = +\ 43.9$ " "

" " just below joint $\quad M_1 = m_1 - b_1X_1 - b_1X_2 - b_1X_3 - = \frac{H}{2}(h_3 - h_1) - b_1(X_1 + X_2 + X_3) = -\ 24.9$ " "

" $x = 0$, at base of post $\quad M_B = m - bX_1 - bX_2 - bX_3 = \frac{H}{2} \times h_3 - b(X_1 + X_2 + X_3) = +\ 58.5$ " "

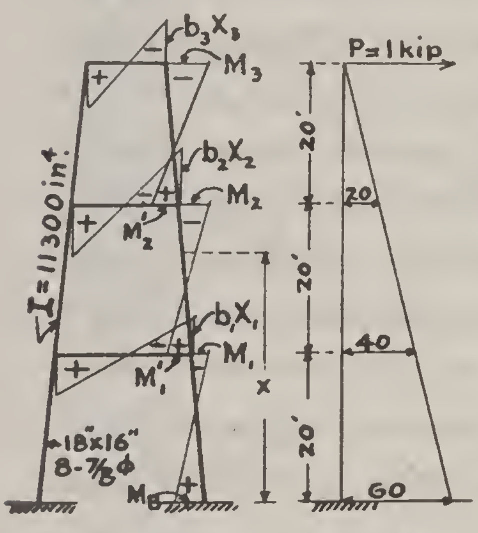
The moment on the top beam, next to the post $= b_3X_3 = -\ 82.0$ kip ft

" " " " 2nd " " " " " $= b_2X_2 = -\ 99.68$ " "

" " " " bottom " " " " " $= b_1X_1 = -\ 68.76$ " "

As a test, the sum of the moments above and below a joint, must equal the beam moment. Thus - $b_2X_2 = M_2' + M_2 = 99.68 = 45.2 + 54.7$ nearly. Also $b_1X_1 = M_1' + M_1 = 68.76 = 43.9 + 24.9$ nearly.

Problem 29G Find the horizontal Deflection δ_3 at the top of the bent for H=16 kips.



M_x Diagram. $\qquad$ M_a Area.

$I = 11300 \text{ in}^4 = 0.555 \text{ ft}^4$.

E = 504,000 kips p.sq.ft for Concrete.

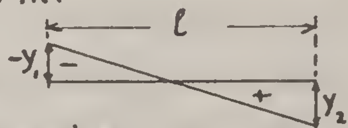
The adjoining M_x diagram shows the moments at all points of the post, as just found in the previous Problem 29E. These moments cause the deflection of the post and each post gets H/2 horizontal load. Hence the work Eq. 14d may be used to find the deflection. We will evaluate $EI\delta_3 = \int M_x M_a dx$ from Table 26A.

The M_a diagram is drawn for a horizontal force P = 1 kip, and the integration is evaluated from Case 4, in 3 lengths of 20 ft. Thus:

$$EI\delta_3 = \tfrac{20}{6}\left[20(2M_2' - M_3)\right] + \tfrac{20}{6}\left[40(2M_1' - M_2) + 20(M_1' - 2M_2)\right] +$$
$$\tfrac{20}{6}\left[60(2M_B - M_1) + 40(M_B - 2M_1)\right] = \tfrac{20}{6}\left[168 + 14 + 5874\right] = 20{,}192$$

Hence, $\delta_3 = \dfrac{20{,}192}{504{,}000 \times 0.555} = 0.0723 \text{ ft.} = 0.867$ in.

It should be noted that a figure like



is merely a trapazoid with one end ordinate negative.

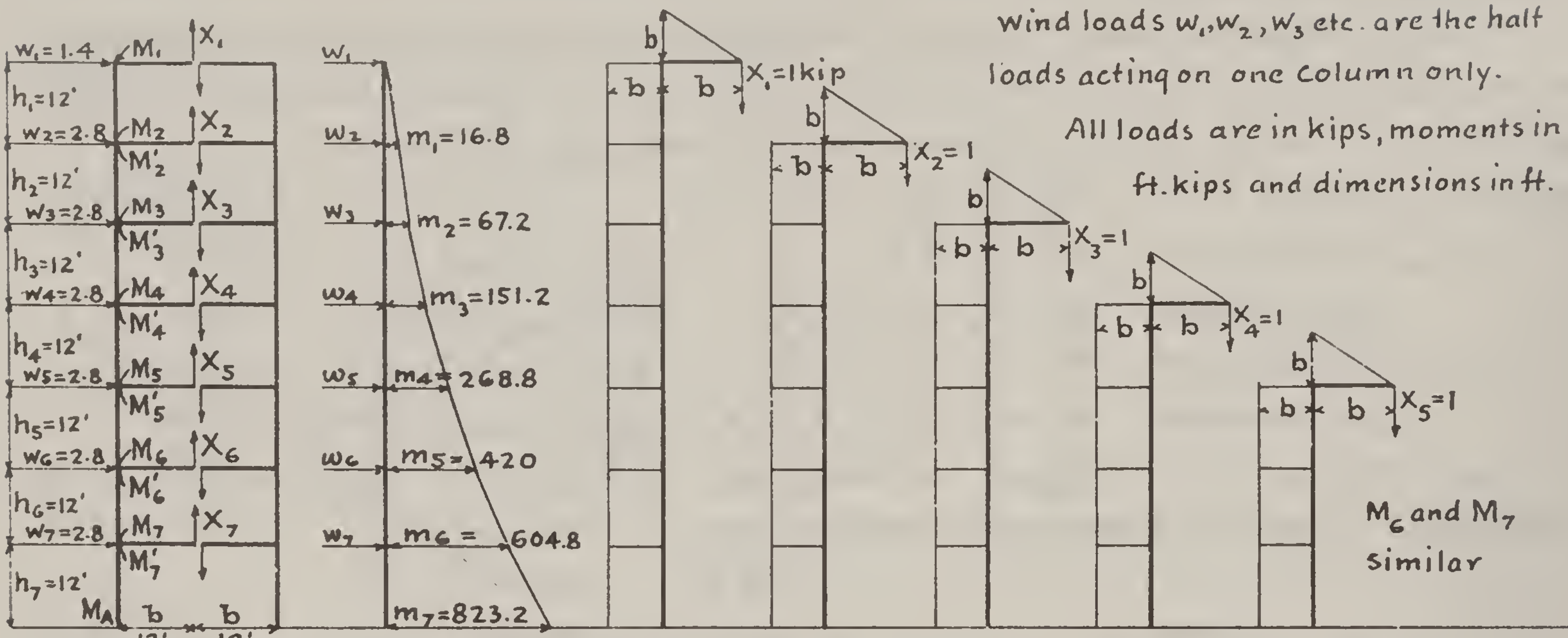
While it was unnecessary to introduce the actual values for I in solving for moments in Problem 29E, yet when deflections are to be found it becomes imperative to use the actual value of I, observing also the units of length employed in solving the moment problem.

If loads H had been applied at each story of the bent, that would merely alter the M_o Area in Problem 29E, and add slightly to the numerical work of finding the final M_x diagram.

Problem 29H. Seven story Bent, subjected to horizontal wind Loads.

Assumptions as for the three story bent, Problem 29 F, constant I.

Wind loads w_1, w_2, w_3 etc. are the half loads acting on one column only. All loads are in kips, moments in ft.kips and dimensions in ft.



Principal system. M_0 area M_1 area M_2 area M_3 area M_4 area M_5 area etc.

The elasticity equations, according to Eqs. 19C, for constant EI are as follows:

$$X_1\int M_1^2dx + X_2\int M_1M_2dx + X_3\int M_1M_3dx + X_4\int M_1M_4dx + X_5\int M_1M_5dx + X_6\int M_1M_6dx + X_7\int M_1M_7dx = \int M_0M_1dx.$$

$$X_1\int M_2M_1dx + X_2\int M_2^2dx + X_3\int M_2M_3dx + X_4\int M_2M_4dx + X_5\int M_2M_5dx + X_6\int M_2M_6dx + X_7\int M_2M_7dx = \int M_0M_2dx.$$

$$X_1\int M_3M_1dx + X_2\int M_3M_2dx + X_3\int M_3^2dx + X_4\int M_3M_4dx + X_5\int M_3M_5dx + X_6\int M_3M_6dx + X_7\int M_3M_7dx = \int M_0M_3dx.$$

$$X_1\int M_4M_1dx + X_2\int M_4M_2dx + X_3\int M_4M_3dx + X_4\int M_4^2dx + X_5\int M_4M_5dx + X_6\int M_4M_6dx + X_7\int M_4M_7dx = \int M_0M_4dx$$

$$X_1\int M_5M_1dx + X_2\int M_5M_2dx + X_3\int M_5M_3dx + X_4\int M_5M_4dx + X_5\int M_5^2dx + X_6\int M_5M_6dx + X_7\int M_5M_7dx = \int M_0M_5dx$$

$$X_1\int M_6M_1dx + X_2\int M_6M_2dx + X_3\int M_6M_3dx + X_4\int M_6M_4dx + X_5\int M_6M_5dx + X_6\int M_6^2dx + X_7\int M_6M_7dx = \int M_0M_6dx$$

$$X_1\int M_7M_1dx + X_2\int M_7M_2dx + X_3\int M_7M_3dx + X_4\int M_7M_4dx + X_5\int M_7M_5dx + X_6\int M_7M_6dx + X_7\int M_7^2dx = \int M_0M_7dx$$

All these integrals are evaluated as per Case 4, Table 26A, noting that those consisting of like subscripts as $\int M_2M_4dx$ and $\int M_4M_2dx$, are equal by Maxwell's Law. For the first $\int M_0M_1dx$, we get $= \frac{b}{2}[h_1m_1 + h_2(m_1+m_2) + h_3(m_2+m_3) + h_4(m_3+m_4) + h_5(m_4+m_5) + h_6(m_5+m_6) + h_7(m_6+m_7)]$.

For equal story heights this gives, after collecting terms, $\int M_0M_1dx = bh[\Sigma_1^6 m + \frac{1}{2}m_7] = 279,420$.

Similarly find all the above integrals evaluated as follows:

$\int M_0M_1dx = bh[\Sigma_1^6 m + \frac{1}{2}m_7] = 279,420$

$\int M_0M_2dx = bh[\Sigma_2^6 m + \frac{1}{2}(m_1+m_7)] = 278,210$

$\int M_0M_3dx = bh[\Sigma_3^6 m + \frac{1}{2}(m_2+m_7)] = 272,162$

$\int M_0M_4dx = bh[\Sigma_4^6 m + \frac{1}{2}(m_3+m_7)] = 256,438$

$\int M_0M_5dx = bh[\Sigma_5^6 m + \frac{1}{2}(m_4+m_7)] = 226,198$

$\int M_0M_6dx = bh[m_6 + \frac{1}{2}(m_5+m_7)] = 176,604$

$\int M_0M_7dx = bh[\frac{1}{2}(m_6+m_7)] = 102,816$

$\int M_1^2dx = b^2[\Sigma_1^7 h + \frac{b}{3}] = 12,672$

$\int M_1M_2dx = b^2\Sigma_2^7 h$ ..... $= 10,368$

$\int M_1M_3dx = b^2\Sigma_3^7 h$ ...... $= 8,640$

$\int M_1M_4dx = b^2\Sigma_4^7 h$ ...... $= 6,912$

$\int M_1M_5dx = b^2\Sigma_5^7 h$ ...... $= 5,184$

$\int M_1M_6dx = b^2\Sigma_6^7 h$ .... $= 3,456$

$\int M_1M_7 = b^2h$ .... $= 1,728$

$\int M_2^2dx = b^2[\Sigma_2^7 h + \frac{b}{3}] = 10,944$

$\int M_2M_3dx = b^2\Sigma_3^7 h$ ... $= 8,640$

$\int M_2M_4dx = b^2\Sigma_4^7 h$ ... $= 6,912$

$\int M_2M_5dx = b^2\Sigma_5^7 h$ ... $= 5,184$

$\int M_2M_6dx = b^2\Sigma_6^7 h$ ... $= 3,456$

$\int M_2M_7dx = b^2h$ .... $= 1,728$

$\int M_3^2dx = b^2[\Sigma_3^7 h + \frac{b}{3}] = 9,216$

$\int M_3M_4dx = b^2\Sigma_4^7 h$ ... $= 6,912$

$\int M_3M_5dx = b^2\Sigma_5^7 h$ .... $= 5,184$

$\int M_3M_6dx = b^2\Sigma_6^7 h$ ... $= 3,456$

$\int M_3M_7dx = b^2h$ ..... $= 1,728$

$\int M_4^2dx = 7,488$

$\int M_4M_5dx = 5,184$

$\int M_4M_6dx = 3,456$

$\int M_4M_7dx = 1,728$

$\int M_5^2dx = 5,760$

$\int M_5M_6dx = 3,456$

$\int M_5M_7dx = 1,728$

$\int M_6^2dx = 4,032$

$\int M_6M_7dx = 1,728$

$\int M_7^2dx = 2,304$

These numerical values inserted into the above seven elasticity equations, afford a solution for the beam shears X_1 to X_7. See Tables 29H and 29J for the solution.

TABLE 29H. SOLUTION OF THE ELASTICITY EQUATIONS.

OPERATIONS	COEFFICIENTS OF THE X'S							Numerical Term
	X_1	X_2	X_3	X_4	X_5	X_6	X_7	K
Eq (1)	12,672	10,368	8,640	6,912	5,184	3,456	1,728	279,420
Eq (2)	10,368	10,944	8,640	6,912	5,184	3,456	1,728	278,210
$\frac{10368}{12672}$[Eq(1)]		8,500	7,070	5,660	4,245	2,829	1,415	228,800
Eq $(2)_1$		2,444	1,570	1,252	939	627	313	49,410
Eq (3)	8,640	8,640	9,216	6,912	5,184	3,456	1,728	272,162
$\frac{8640}{12672}$[Eq(1)]		7,070	5,890	4,715	3,535	2,355	1,177	190,500
Eq $(3)_1$		1,570	3,326	2,197	1,649	1,101	551	81,662
$\frac{1570}{2444}$[Eq$(2)_1$]			1,007	806	604	402	201	31,700
Eq $(3)_2$			2,319	1,391	1,045	699	350	49,962
Eq (4)	6,912	6,912	6,912	7,488	5,184	3,456	1,728	256,438
$\frac{6912}{12672}$[Eq(1)]		5,660	4,715	3,778	2,828	1,885	944	152,300
Eq $(4)_1$		1,252	2,197	3,710	2,356	1,571	784	104,138
$\frac{1252}{2444}$[Eq$(2)_1$]			806	643	482	322	160.5	25,320
Eq $(4)_2$			1,391	3,067	1,874	1,249	623.5	78,818
$\frac{1391}{2319}${Eq$(3)_2$]				837	628	420	210.2	30,000
Eq $(4)_3$				2,230	1,246	829	413.3	48,818
Eq (5)	5,184	5,184	5,184	5,184	5,760	3,456	1,728	226,198
$\frac{5184}{12672}$[Eq(1)]		4,245	3,535	2,828	2,120	1,413	708	114,100
Eq $(5)_1$		939	1,649	2,356	3,640	2,043	1,020	112,098
$\frac{939}{2444}${Eq$(2)_1$]			604	482	361	241	120	19,000
Eq $(5)_2$			1,045	1,874	3,279	1,802	900	93,098
$\frac{1045}{2319}$[Eq$(3)_2$]				628	471	315	157.6	22,500
Eq $(5)_3$				1,246	2,808	1,487	742.4	70,598
$\frac{1246}{2230}$[Eq$(4)_3$]					696	463	230.8	27,300
Eq $(5)_4$					2,112	1,024	511.6	43,298
Eq (6)	3,456	3,456	3,456	3,456	3,456	4,032	1,728	176,604
$\frac{3456}{12672}${Eq(1)]		2,829	2,355	1,885	1,413	942	472	76,200
Eq $(6)_1$		627	1,101	1,571	2,043	3,090	1,256	100,404
$\frac{627}{2444}${Eq$(2)_1$]			402	322	241	161	80.3	12,670
Eq $(6)_2$			699	1,249	1,802	2,929	1,175.7	87,734
$\frac{699}{2319}$[Eq$(3)_2$]				420	315	210.2	105.5	15,080
Eq $(6)_3$				829	1,487	2,718.8	1,070.2	72,654
$\frac{829}{2230}$[Eq$(4)_3$]					463	304.0	153.2	18,150
Eq $(6)_4$					1,024	2,414.8	917.0	54,504
$\frac{1024}{2112}$[Eq$(5)_4$]						497.0	248.0	21,000
Eq $(6)_5$						1,917.8	669.0	33,504

TABLE 29H - CONTINUED.

OPERATIONS	COEFFICIENTS OF THE X's							Numerical Term
	X_1	X_2	X_3	X_4	X_5	X_6	X_7	K
Eq(7)	1,728	1,728	1,728	1,728	1,728	1,728	2,304	102,816
$\frac{1728}{1267.2}$[Eq(1)]		1,415	1,177	944	708	472	236	38,160
$Eq(7)_1$		313	551	784	1,020	1,256	2,068	64,656
$\frac{313}{2444}$[$Eq(2)_1$]			201	160.5	120	80.3	40.1	6,330
$Eq(7)_2$			350	623.5	900	1,175.7	2,027.9	58,326
$\frac{350}{2319}$[$Eq(3)_2$]				210.2	157.6	105.5	52.9	7,550
$Eq(7)_3$				413.3	742.4	1,070.2	1,975.0	50,776
$\frac{413.3}{2230}$[$Eq(4)_3$]					230.8	153.2	76.6	9,050
$Eq(7)_4$					511.6	917.0	1,898.4	41,726
$\frac{511.6}{2112}$[$Eq(5)_4$]						248.0	123.4	10,470
$Eq(7)_5$						669.0	1,775.0	31,256
$\frac{669}{1917.8}$[$Eq(6)_5$]							233.2	11,680
$Eq(7)_6$							1,531.8	19,576

$X_7 = 12.75$ kips.

TABLE 29J. EVALUATION OF THE X's BY SUCCESSIVE SUBSTITUTIONS IN LAST SET.

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	
K =	102,816	64,656	58,326	50,776	41,726	31,256	19,576	
C_7X_7	−29,400	−26,400	−25,860	−25,200	−24,200	−22,600	$X_7 = \frac{19576}{1531.8} = 12.75$ kips.	
C_6X_6	−22,340	−16,220	−15,200	−13,840	−11,820	−22,600		
C_5X_5	−19,280	−11,390	−10,030	− 8,290	−36,020	+ 8,656 $X_6 = \frac{8656}{669} = 12.92$		
C_4X_4	−14,400	− 6,540	− 5,200	−47,330	+ 5,706 $X_5 = \frac{5706}{511.6} = 11.16$			
C_3X_3	−10,020	− 3,198	−56,290	+ 3,446 $X_4 = \frac{3446}{413.3} = 8.35$				
C_2X_2	− 5,015	−63,748	+ 2,036 $X_3 = \frac{2036}{350} = 5.81$					
The coefficients C are from the last set of Tables 29H	−100,455 + 2,361 $X_1 = \frac{2361}{1728} = 1.365$	+ 908 $X_2 = \frac{908}{313} = 2.90$						

$\Sigma_1^7 X = 55.26$

Values satisfy Eq(1) to 1/4 %

BEAM MOMENTS.

Roof	$12X_1 = 16.38$ ft. kips.	
7th Floor	$12X_2 = 34.8$	
6th ..	$12X_3 = 69.7$	
5th ..	$12X_4 = 120.4$	
4th ..	$12X_5 = 133.9$	
3rd ..	$12X_6 = 155.0$	
2nd ..	$12X_7 = 153.0$	

COLUMN MOMENTS.

$M_1 = bX_1 = 12 \times 1.365$ ― ― ― ― ― $= +16.38$ ft. kips.

$M_2 = m_1 - bX_1 = 16.8 - 16.38$ ― ― ― ― $= +0.42$

$M_2' = m_1 - b(X_1 + X_2) = 16.8 - 51.2$ ― ― ― $= -34.4$

$M_3 = m_2 - b(X_1 + X_2) = 67.2 - 51.2$ ― ― ― $= +16.0$

$M_3' = m_2 - b(X_1 + X_2 + X_3) = 67.2 - 120.9 = -53.7$

$M_4 = m_3 - b(X_1 + X_2 + X_3) = 151.2 - 120.9 = +30.3$

$M_4' = m_3 - b\Sigma_1^4 X = 151.2 - 241.3 = -90.1$

$M_5 = m_4 - b\Sigma_1^4 X = 268.8 - 241.3 = +27.5$

$M_5' = m_4 - b\Sigma_1^5 X = 268.8 - 375.3 = -106.5$

$M_6 = m_5 - b\Sigma_1^5 X = 420.0 - 375.3 = +44.7$

$M_6' = m_5 - b\Sigma_1^6 X = 420.0 - 530.3 = -110.3$

$M_7 = m_6 - b\Sigma_1^6 X = 604.8 - 530.3 = +74.5$

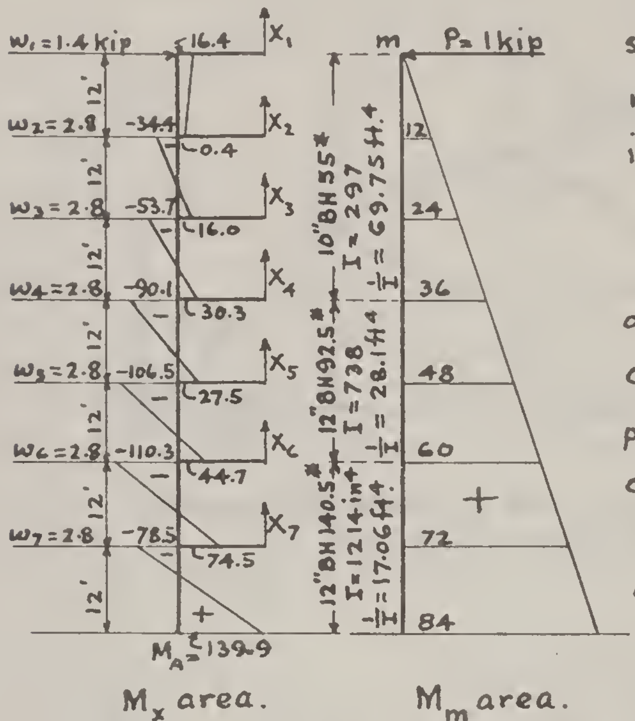
$M_7' = m_6 - b\Sigma_1^7 X = 604.8 - 683.3 = -78.5$

$M_A = m_7 - b\Sigma_1^7 X = 823.2 - 683.3 = +139.9$

Problem 29K. Find the Horizontal Deflection at the top of the Seven Story Bent, Problem 29H.

The M_x area represents the actual moments in the column, the deflection of which is to be found for the top point m. The beam moments have been omitted, since their effects are already represented in the column moments. The problem is solved in the manner illustrated in Problem 29G, employing Mohr's Work Equation, whence

$$\delta_m = \frac{h}{EI}\int_0^l M_x M_m dx .$$



M_x area. M_m area.

The M_m diagram is drawn for P = 1 kip, acting horizontally at the point m, and the integrations are evaluated from Case 4, Table 26A, in seven lengths of 12 ft., observing positive and negative moments. The values $\frac{1}{I}$ in.^{4} must be converted into $\frac{12^4}{I}$ ft.^{4}, and E = 4,320,000 kips per sq. ft.

The values $\frac{1}{I}$ ft.^{4} for the three column sections are written on the M_m diagram.

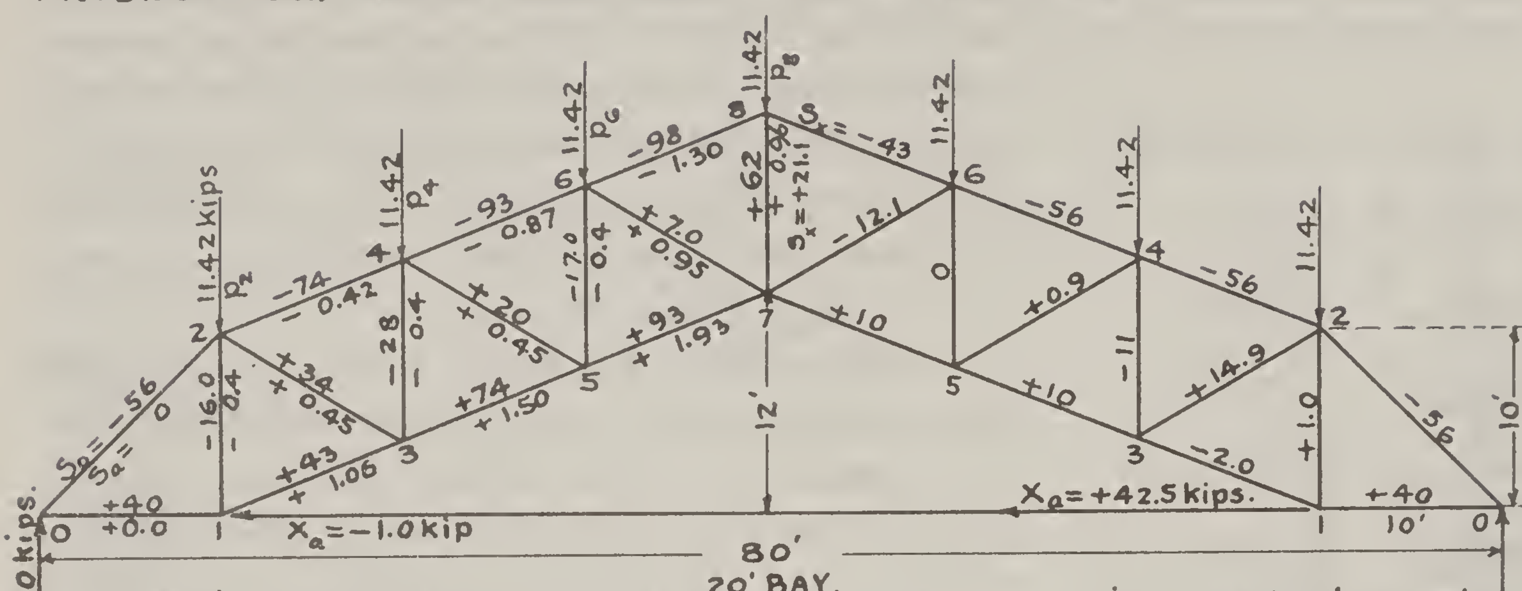
The substitution formula, Case 4, Table 26A, is $\frac{h}{I}\int M_z M_y dx = \frac{h}{6I}\left[z_1(2y_1+y_2) + z_2(y_1+2y_2)\right]$, in which the z_1 and z_2 ordinates may be the M_x moments, and the y_1 and y_2 ordinates are the M_m moments.

This gives the seven values as follows, in units of feet and kips.

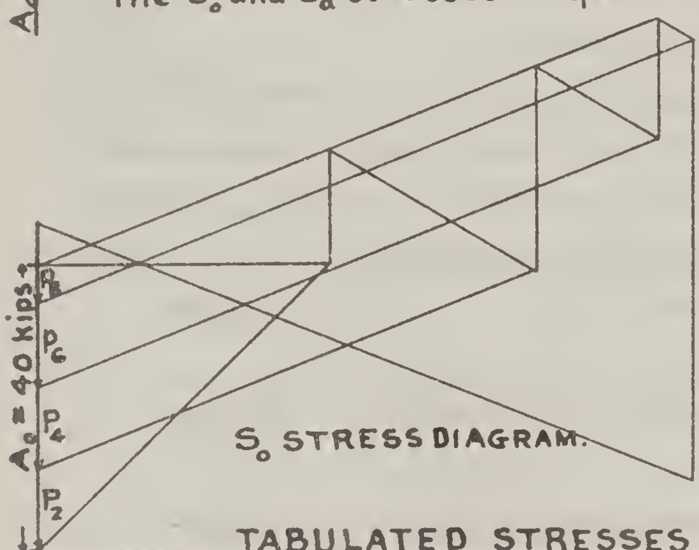
Top or 7th story	$\frac{h}{I}\int M_x M_m dx$	$= \frac{12\times69.75}{6}\left[16.4\times12 + 0.4\times2\times12\right]$	$= +\ 28,800$
6th story	"	$= \frac{12\times69.75}{6}\left[-34.4(2\times12+24) + 16(12+2\times24)\right]$	$= -\ 96,800$
5th story	"	$= \frac{12\times69.75}{6}\left[-53.7(2\times24+36) + 30.3(24+2\times36)\right]$	$= -\ 225,000$
4th story	"	$= \frac{12\times28.1}{6}\left[-90.1(2\times36+48) + 27.5(36+2\times48)\right]$	$= -\ 403,000$
3rd story	"	$= \frac{12\times28.1}{6}\left[-106.5(2\times48+60) + 44.7(48+2\times60)\right]$	$= -\ 511,500$
2nd story	"	$= \frac{12\times17.06}{6}\left[-110.3(2\times60+72) + 74.5(60+2\times72)\right]$	$= -\ 198,000$
1st story	"	$= \frac{12\times17.06}{6}\left[-78.5(2\times72+84) + 139.9(72+2\times84)\right]$	$= +\ 535,000$
		Total $\frac{h}{I}\int M_x M_m dx$	$= -\ 870,500$

Hence $\delta_m = -\frac{870\,500}{E} = -\frac{870\,500}{4,320,000} = -0.201$ ft. $= -2.42$ in., indicating movement to the right.

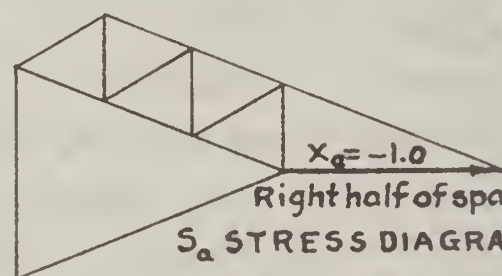
PROBLEM 29L: TWO-HINGED ARCH WITH HORIZONTAL TIE.



The principal system is a simple span on two supports, with X_a = redundant member. The S_o and S_a stresses are given on the left half, and S_x on the right half.



S_o STRESS DIAGRAM.



Right half of span.
S_a STRESS DIAGRAM.

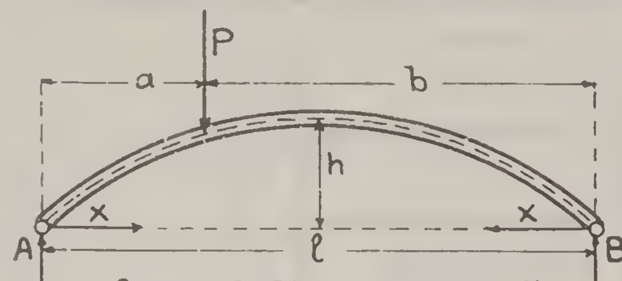
Tie member 2L^s $3 \times 2\frac{1}{2} \times \frac{1}{4}$, 60 ft. long. $\frac{l_a}{F_a} = 275$.
Table below gives $\Sigma S_o S_a \, l/F = 69032$, $\Sigma S_a^2 \, l/F = 1350$

$$\text{Then } X_a = \frac{\Sigma S_o S_a \, l/F}{\Sigma S_a^2 \, l/F + l_a/F_a} = \frac{69032}{1350+275} = 42.5 \text{ kips.}$$

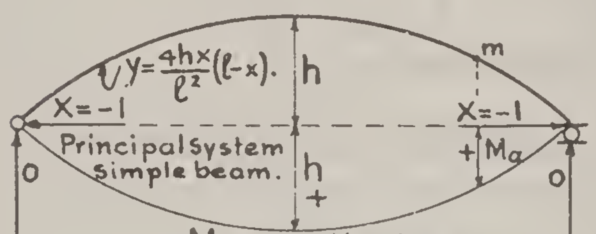
TABULATED STRESSES S_o, S_a AND $S_x = S_o - S_a X_a$. ALL IN KIPS.

Member	S_o		S_a		$S_o S_a$	S_a^2	Section	Gross Area F sq. in.	Length l in	l/F	$S_o S_a \frac{l}{F}$	$S_a^2 \frac{l}{F}$	$S_x = S_o - S_a X_a$
	+	−	+	−	+	+							
0-1	40	-	0	-	0	0	2L $2\frac{1}{2} \times 2 \times \frac{3}{16}$	1.62	120	74	0	0	+ 40
1-3	43	-	1.06	-	45.6	1.12	2L $2\frac{1}{2} \times 2\frac{1}{2} \times \frac{3}{16}$	1.80	129	72	3270	81	− 2
3-5	74	-	1.50	-	111.	2.25	"	1.80	129	72	8000	161	+ 10
5-7	93	-	1.95	-	181	3.80	"	1.80	129	72	13000	272	+ 10
0-2	-	56	-	0	0	0	2L $4 \times 3 \times 5/16$	4.18	168	40.3	0	0	− 56
2-4	-	74	-	0.42	31.1	0.18	2L $4 \times 3 \times 5/16$ 1 plt. $12 \times 1/4$	7.18	129	18	558	3	− 56.2
4-6	-	93	-	0.87	80.8	0.76	"	7.18	129	18	1450	14	− 56
6-8	-	98	-	1.30	127.2	1.69	"	7.18	.129	18	2290	30.5	− 43
1-2	-	16	-	0.40	6.4	0.16	2L $2\frac{1}{2} \times 2 \times \frac{3}{16}$	1.62	120	74	473	11.8	+ 1
3-4	-	28	-	0.40	11.2	0.16	2L $3 \times 3 \times \frac{1}{4}$	2.88	1.20	41.7	468	6.7	− 11.
5-6	-	17	-	0.40	6.8	0.16	2L $2\frac{1}{2} \times 2 \times \frac{3}{16}$	1.62	1.20	74	503	11.8	0
2-3	34	-	0.45	-	15.3	0.20	"	1.62	138	85	1300	17	+ 14.9
4-5	20	-	0.45	-	9.0	0.20	"	1.62	138	85	765	17	+ 0.9
6-7	7	-	0.45	-	3.1	0.20	2L $2\frac{1}{2} \times 2\frac{1}{2} \times \frac{3}{16}$	1.80	138	77	239	15.4	− 12.1
										Sum	32316 32316	641.2 641.2	
7-8	62	-	0.96	-	59.5	0.92	2L $2\frac{1}{2} \times 2 \times \frac{3}{16}$	1.62	120	74	4400	68.0	+ 21.1
										Σ =	69032	1350.4	

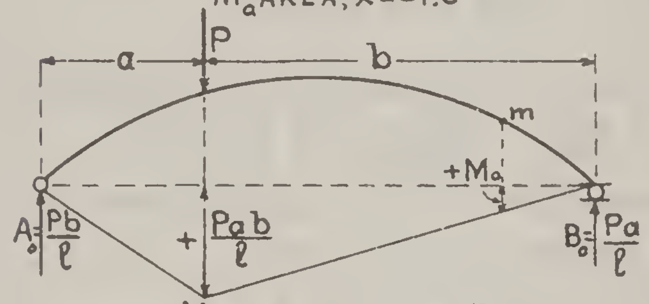
PROBLEM 29-M. TWO-HINGED PARABOLIC ARCH RIB. CONSTANT EI.



GIVEN CASE, HOR. THRUST X.

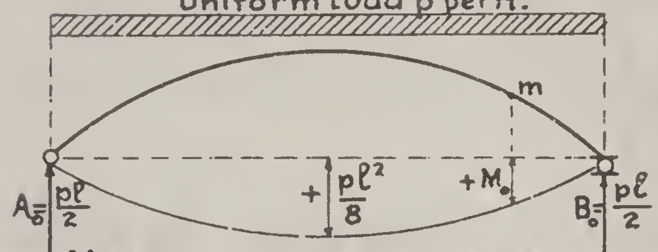


M_a AREA, X=-1.0

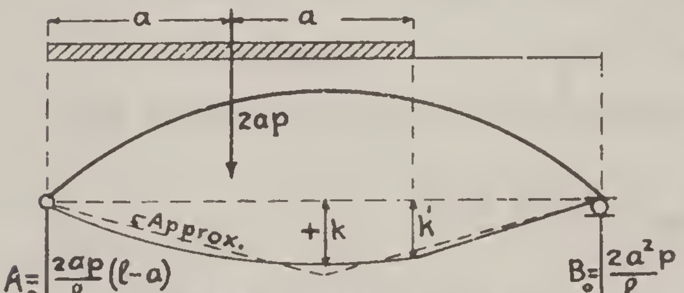


M_o AREA FOR LOAD P.

Uniform Load p per ft.



M_o AREA, LOAD OVER WHOLE SPAN.



M_o AREA FOR PARTIAL LOAD.

$k=p\left(a\ell-a^2-\frac{\ell^2}{8}\right)$; $k'=2pa^2\left(1-\frac{2a}{\ell}\right)$.

The problem involves one redundant X, which represents the arch thrust, and which when removed, converts the structure into a simple beam as the principal system. According to Eq. 19-C, then for constant EI and neglecting axial thrust, the redundant X is found from

$1.\delta=\frac{1}{EI}\int M_o M_a dx - \frac{X}{EI}\int M_a^2 dx = 0$, giving the value $X=\frac{\int M_o M_a dx}{\int M_a^2 dx}$ ---------------(1)

With X known, Eq. 19B gives the moment at any point m as $M_m = M_o - M_a X$ -------------(2)

The M_a area for X=-1, is independent of the live loads and hence $\int M_a^2 dx$ is a constant depending only on the coordinates of the arch axis.

From Table 26-A, case 6, find

$$\int M_a^2 dx = \frac{8}{15}\ell h^2 \text{ ----------------------(3)}$$

The quantity $\int M_o M_a dx$ depends on the loads and will be evaluated for the following cases:

For a single load P, Table 26-A, case 7, gives

$$\int M_o M_a dx = \frac{Pabh}{3\ell}\left[\ell+\frac{ab}{\ell}\right] = \frac{Pabh}{3}\left(1+\frac{ab}{\ell^2}\right),$$

hence $X=\frac{\int M_o M_a dx}{\int M_a^2 dx} = \frac{5}{8}\frac{Pab}{\ell h}\left(1+\frac{ab}{\ell^2}\right)$ ---- (4)

For several loads P, the formula for X may be written as

$$X=\frac{5}{8\ell h}\Sigma Pab\left(1+\frac{ab}{\ell^2}\right) \text{ -------------(5)}$$

For a uniform load p per ft. over the whole span, Table 26-A, case 12, gives

$$\int M_o M_a dx = \frac{8}{15}\ell h \frac{p\ell^2}{8} = \frac{1}{15} p h \ell^3$$

hence $X=\frac{p\ell^2}{8h}$ ------------------(6)

For a uniform load over a distance 2a, it is best to evaluate $\int M_o M_a dx$ numerically by Case 7, using an approximate triangular M_o area.

When X is found for any given case of loading, then $M_m = M_o - M_a X$.

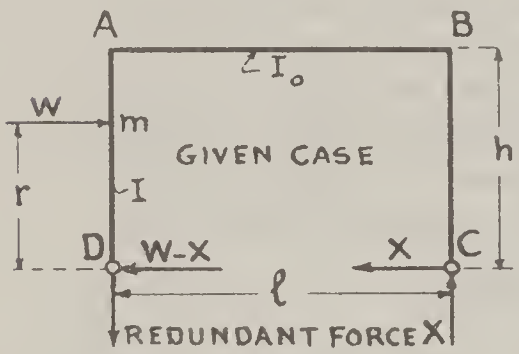
Temperature effect. The horizontal thrust X_t, due to a rise in temperature of t° F., is found as follows: For a coefficient of expansion ε, the change in the length ℓ is $\Delta\ell = \varepsilon t \ell$, making the temperature thrust –

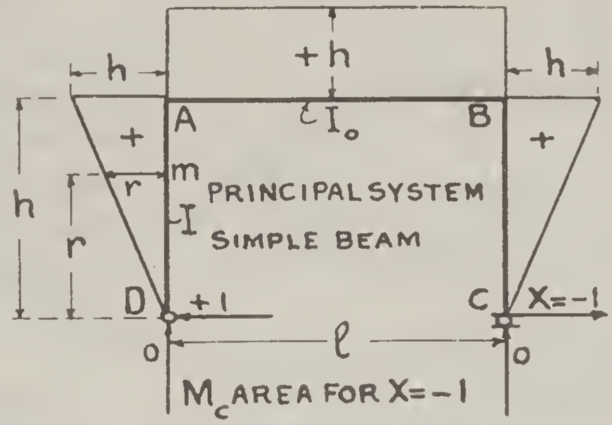
$$X_t = \frac{\varepsilon t \ell}{\frac{1}{EI}\int M_a^2 dx} = \frac{15\,EI}{8h^2}\varepsilon t \text{ ---------------------------(7)}$$

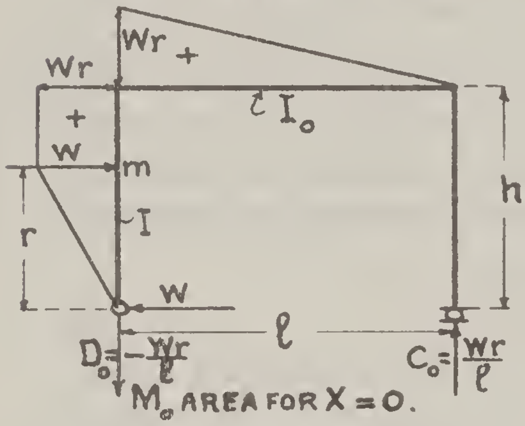
and the moment, at any axial point m, is $M_{mt} = -M_a X_t$.

For an exhaustive treatment of the subject of arches, see the author's Kinetic Theory of Engineering Structures, 1911.

PROBLEM 29-N. TWO-HINGED RECTANGULAR FRAME, LOAD W ON POST.







As in Problem 29-M, the general formulas for the case of one redundant condition are

$$X = \frac{\frac{1}{I}\int M_o M_c\,dx}{\frac{1}{I}\int M_c^2\,dx} \qquad \text{and} \quad M_m = M_o - M_c X.$$

Using the substitution formulas 1 and 2, Table 26-A,

$$\frac{1}{I}\int M_c^2\,dx = \frac{2h^3}{3I} + \frac{\ell h^2}{I_o} = h^2\left[\frac{2h}{3I} + \frac{\ell}{I_o}\right] \text{ ------ } A$$

which is independent of the loading and applies to all two-hinged rectangular frames.

According to the formulas 2, 3 and 4, Table 26-A, find

$$\frac{1}{I}\int M_o M_c\,dx = \frac{rWr}{6I}\times 2r + \frac{h-r}{6I}\Big[Wr(2r+h) + Wr(r+2h)\Big] + \frac{\ell Wr}{6I_o}\times 3h = \left[h^2 - \frac{r^2}{3} + \frac{h\ell I}{I_o}\right]\frac{Wr}{2I}.$$

Hence
$$X = \frac{\left[h^2 - \frac{r^2}{3} + h\ell\frac{I}{I_o}\right]\frac{Wr}{2I}}{h^2\left[\frac{2h}{3I} + \frac{\ell}{I_o}\right]} = \left[\frac{h^2 - \frac{r^2}{3} + \frac{h\ell I}{I_o}}{2h^2\left[\frac{2}{3}h + \frac{\ell I}{I_o}\right]}\right] Wr.$$

The moment formula gives $M_m = M_o - M_c X = Wr - rX$, also $M_A = Wr - hX$ and $M_B = -hX$, The final M_m diagram can now be drawn to show the moment at every point.

The reactions are $W-X$, X, $-D = +C = \frac{Wr}{\ell}$.

The change due to temperature in the length ℓ is $\varepsilon t\ell$, for a rise of t°F.,

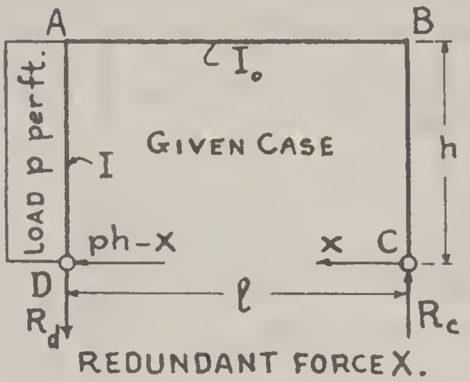
giving
$$X_t = \frac{\varepsilon t\ell}{\frac{h^2}{EI}\left(\frac{2}{3}h + \frac{\ell I}{I_o}\right)} = \frac{\varepsilon t\ell EI}{h^2\left(\frac{2}{3}h + \frac{\ell I}{I_o}\right)}, \quad \text{and } M_{mt} = -M_c X_t.$$

<u>SPECIAL CASE WHEN W IS APPLIED AT A,</u> then $r = h$ and the above formulas become
$$X = \frac{\frac{2}{3}h^3 + h^2\ell\frac{I}{I_o}}{2\left(\frac{2}{3}h^3 + h^2\ell\frac{I}{I_o}\right)}W = \frac{W}{2}, \quad \text{and} \quad M_A = h(W-X),\ M_B = -hX,\ -D = +C = \frac{Wh}{\ell}.$$

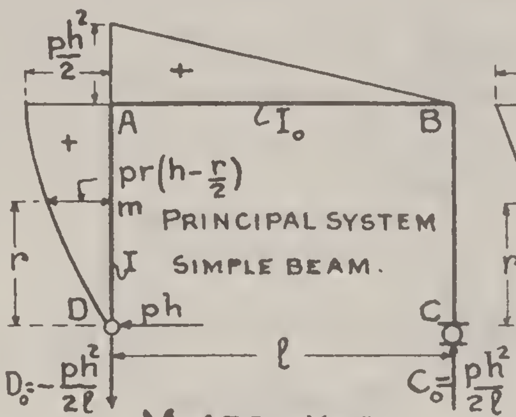
<u>For the following cases of loading</u> on the two-hinged rectangular frame, the integral $\frac{1}{I}\int M_c^2 dx = h^2\left[\frac{2h}{3I} + \frac{\ell}{I_o}\right]$, retains this form and will not be evaluated again except for frames of different shape.

However, in describing the several cases which follow, the M_o and M_c diagrams will be shown for purposes of illustration.

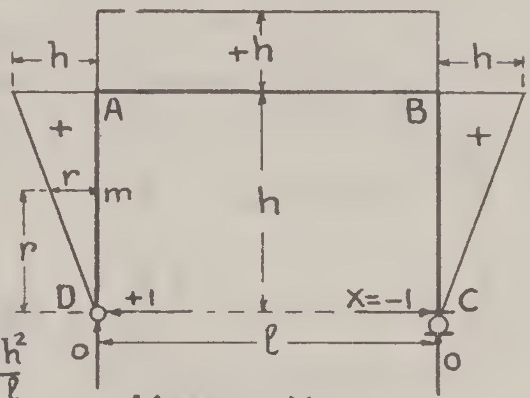
PROBLEM 29-O. TWO-HINGED RECTANGULAR FRAME. UNIFORM POST LOAD.



REDUNDANT FORCE X.



M_o AREA, X=0



M_c AREA, X=−1.

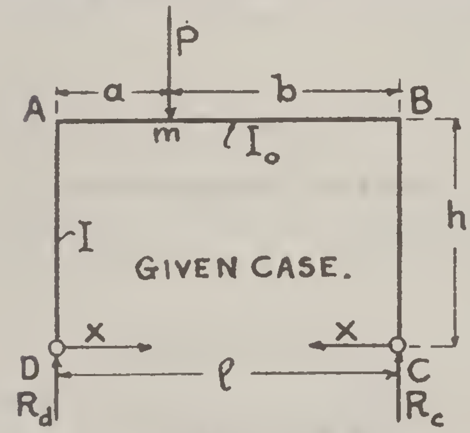
With substitution formulas 3 and 10, Table 26-A, evaluate the integral

$\frac{1}{I}\int M_o M_c dx = \frac{h\times ph^2}{8I}\times\frac{5}{3}h + \frac{\ell ph^2}{12 I_o}\times 3h = \frac{ph^3}{4}\left[\frac{5h}{6I}+\frac{\ell}{I_o}\right]$, which together with the value $\frac{1}{I}\int M_c^2 dx = h^2\left[\frac{2h}{3I}+\frac{\ell}{I_o}\right]$, as found in Problem 29-N, gives

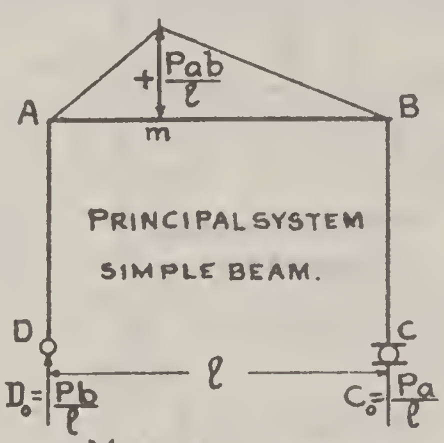
$$X = \frac{\frac{ph^3}{4}\left(\frac{5h}{6I}+\frac{\ell}{I_o}\right)}{h^2\left(\frac{2h}{3I}+\frac{\ell}{I_o}\right)} = \left[\frac{\frac{5h}{6I}+\frac{\ell}{I_o}}{\frac{2h}{3I}+\frac{\ell}{I_o}}\right]\frac{ph}{4} = \left[\frac{5hI_o+6I\ell}{4hI_o+6I\ell}\right]\frac{ph}{4}.$$

From the moment Eq. 19B, find $M_m = M_o - M_c X = pr\left(h-\frac{r}{2}\right) - rX$, $M_A = \frac{ph^2}{2} - hX$, $M_B = 0 - hX$.
The reactions are $ph - X$, X, and $-R_d = +R_c = \frac{ph^2}{2\ell}$. Also $X_t = \frac{\varepsilon t \ell E I}{h^2\left[\frac{2}{3}h + \frac{\ell I}{I_o}\right]}$, and $M_{mt} = -M_c X_t$.

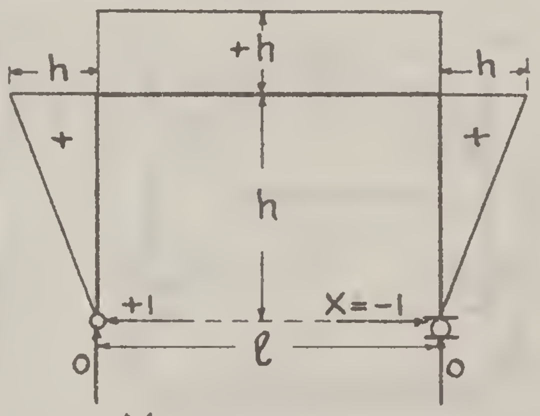
PROBLEM 29-P. TWO-HINGED RECTANGULAR FRAME. LOAD P ON BEAM.



REDUNDANT FORCE X.



M_o AREA, X = 0.



M_c AREA, X=−1.

With substitution formulas 5, Table 26-A, evaluate the integral

$\frac{1}{I}\int M_o M_c dx = \frac{Pab}{6\ell I_o}\left[2h\ell + bh + ah\right] = \frac{Pabh}{2I_o}$. With the above value of $\frac{1}{I}\int M_c^2 dx$,

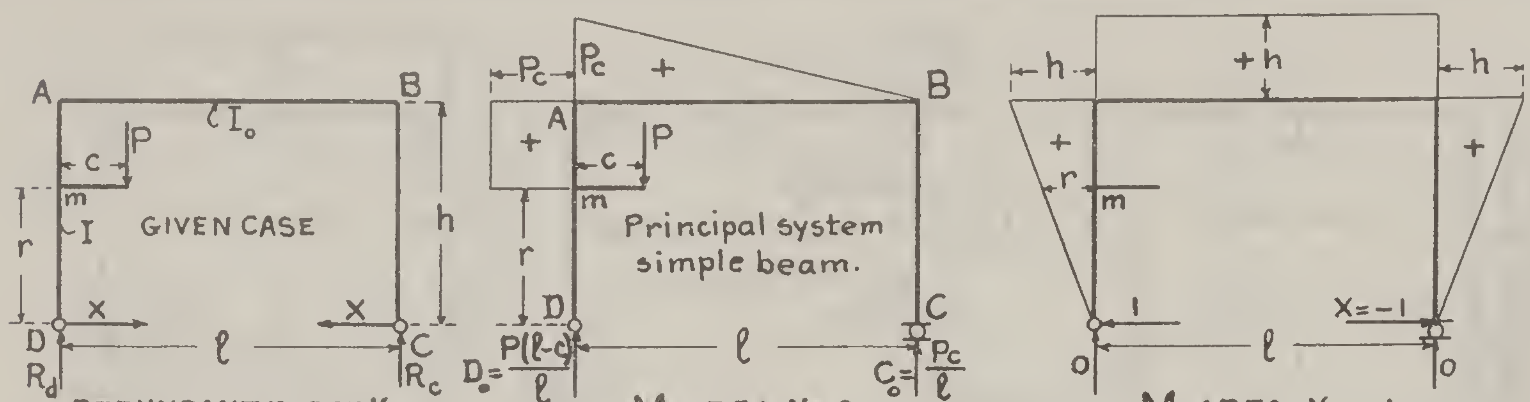
find $X = \frac{Pabh}{2I_o h^2\left(\frac{2h}{3I}+\frac{\ell}{I_o}\right)} = \frac{Pab}{2h\left(\frac{2hI_o}{I}+\ell\right)}$. For several loads P, find $X = \frac{\Sigma Pab}{2h\left(\frac{2hI_o}{I}+\ell\right)}$.

The moment Eq. 19B gives $M_m = M_o - M_c X = \frac{Pab}{\ell} - hX$, also $M_A = M_B = -hX$.

The reactions are X, $R_c = \frac{Pa}{\ell}$ and $R_d = \frac{Pb}{\ell}$, with X_t and M_{mt} as above.

PROBLEM 29-Q. UNIFORM LOAD p per ft. over the beam. The M_o Area becomes a parabola with middle ordinate $\frac{p\ell^2}{8}$, giving $\frac{1}{I}\int M_o M_c dx = \frac{\ell}{3I_o}\times 2h\times\frac{p\ell^2}{8} = \frac{ph\ell^3}{12 I_o}$, and $X = \frac{p\ell^3}{4h\left(\frac{2hI_o}{I}+3\ell\right)}$ making $M_m = \frac{p\ell^2}{8} - hX$, $M_A = M_B = -hX$, and $R_c = R_d = \frac{p\ell}{2}$.

PROBLEM 29-R. TWO-HINGED RECTANGULAR FRAME, CRANE LOAD ON ONE COLUMN.



REDUNDANT FORCE X. $\quad$ M_o AREA, X = 0. $\quad$ M_c AREA, X = −1.

With substitution formula 4, Table 26-A, evaluate the integral

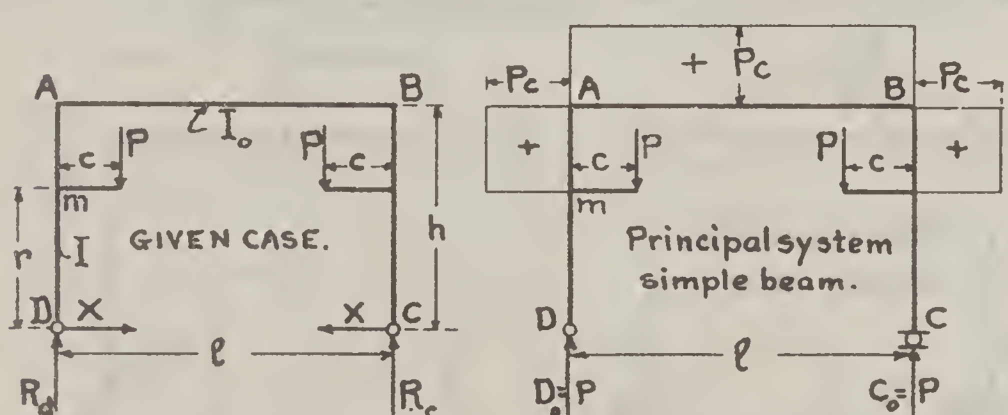
$$\frac{1}{I}\int M_o M_c dx = \frac{h-r}{6I}\left[r\times 3Pc + h\times 3Pc\right] + \frac{\ell}{6I_o}\left[h\times 2Pc + hPc\right] = Pc\left[\frac{h^2-r^2}{2I} + \frac{h\ell}{2I_o}\right].$$

Using the value $\frac{1}{I}\int M_c^2 dx = h^2\left(\frac{2h}{3I} + \frac{\ell}{I_o}\right)$, as found in Problem 29-N, to obtain

$$X = \left[\frac{\frac{h^2-r^2}{2I} + \frac{h\ell}{2I_o}}{h^2\left(\frac{2h}{3I} + \frac{\ell}{I_o}\right)}\right]Pc = \left[\frac{(h^2-r^2)I_o + h\ell I}{2h^2\left(\frac{2}{3}hI_o + \ell I\right)}\right]Pc$$

From the moment Eq. 19-B, find $M_m = 0 - rX$ below, and $M'_m = Pc - rX$ above the bracket. Also $M_A = Pc - hX$, $M_B = -hX$; The reactions are X, $R_d = P\frac{(\ell-c)}{\ell}$, and $R_c = \frac{Pc}{\ell}$.

PROBLEM 29-S. TWO-HINGED RECTANGULAR FRAME, CRANE LOADS BOTH COLS.



SAME AS PROBLEM 29-R.

REDUNDANT FORCE X. $\quad$ M_o AREA, X = 0. $\quad$ M_c AREA FOR X = −1.

With formulas 1 and 3, Table 26-A, evaluate the integral

$$\frac{1}{I}\int M_o M_c dx = \frac{2(h-r)}{6I}\left[r\times 3Pc + h\times 3Pc\right] + \frac{\ell hPc}{I_o} = \left[\frac{h^2-r^2}{I} + \frac{\ell h}{I_o}\right]Pc.$$

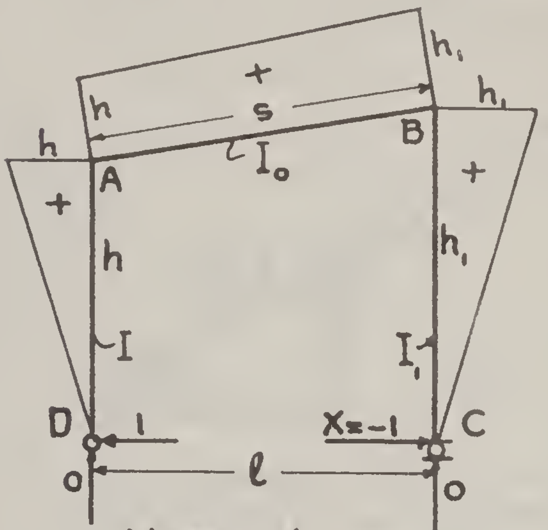
As in Prob. 29-N, find $\frac{1}{I}\int M_c^2 dx = h^2\left(\frac{2h}{3I} + \frac{\ell}{I_o}\right)$, which gives

$$X = \left[\frac{\frac{h^2-r^2}{I} + \frac{\ell h}{I_o}}{h^2\left(\frac{2h}{3I} + \frac{\ell}{I_o}\right)}\right]Pc = \left[\frac{(h^2-r^2)I_o + \ell hI}{h^2\left(\frac{2}{3}hI_o + \ell I\right)}\right]Pc.$$

The moment Eq. 19-B gives $M_m = 0 - rX$ below, and $M'_m = Pc - rX$ above the bracket. Also $M_A = M_B = Pc - hX$. The reactions are X, and $R_c = R_d = P$.

PROBLEM 29-T, VARIOUS TWO-HINGED FRAMES.

The M_o areas for various loadings, for the cases here given, will be similar to those drawn for Problems 29-N to 29-S, and the evaluation of $\int M_o M_c dx$ may be left to the reader for practice. The integral $\int M_c^2 dx$ will however, be given here and each formula remains unchanged for the same frame, irrespective of the loads.



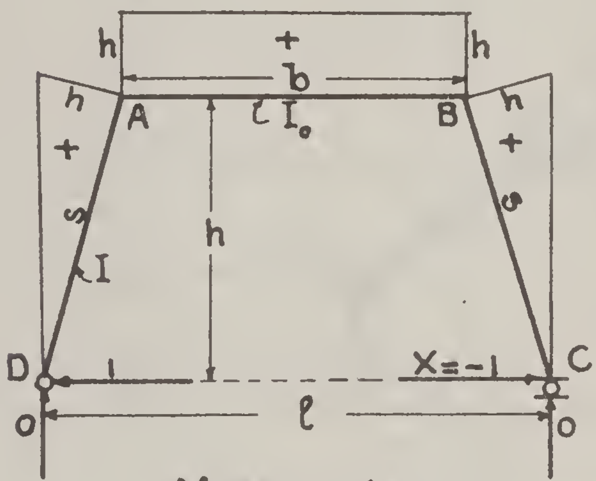
M_c AREA, X=-1.

$$\frac{1}{I}\int M_c^2 dx = \frac{h\times h^2}{3I} + \frac{s}{3I_o}\left[h^2+hh_1+h_1^2\right] + \frac{h_1\times h_1^2}{3I_1} \text{ giving}$$

$$X = \frac{\frac{1}{I}\int M_o M_c dx}{\frac{1}{I}\int M_c^2 dx} = \frac{\frac{1}{I}\int M_o M_c dx}{\frac{h^3}{3I}+\frac{h_1^3}{3I}+\frac{s}{3I_o}\left[h^2+hh_1+h_1^2\right]}.$$

Also $M_m = M_o - M_c X$ and $M_{mt} = -M_c X_t$,

$$X_t = \frac{3\varepsilon t \ell E}{\frac{h^3}{I}+\frac{h_1^3}{I_1}+\frac{s}{I_o}\left[h^2+hh_1+h_1^2\right]}.$$



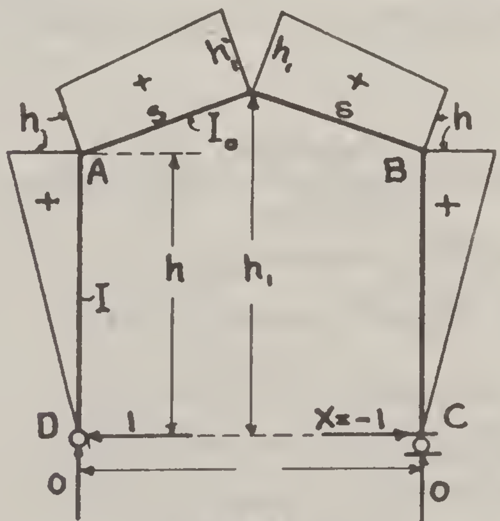
M_c AREA, X=-1.

$$\frac{1}{I}\int M_c^2 dx = \frac{2sh^2}{3I} + \frac{bh^2}{I_o} = h^2\left(\frac{2s}{3I}+\frac{b}{I_o}\right).$$

Note the similarity with Problem 29-N.

$$X = \frac{\frac{1}{I}\int M_o M_c dx}{h^2\left(\frac{2s}{3I}+\frac{b}{I_o}\right)}, \quad M_m = M_o - M_c X.$$

$$X_t = \frac{\varepsilon t \ell E}{h^2\left(\frac{2s}{3I}+\frac{b}{I_o}\right)}, \quad M_{mt} = -M_c X_t.$$



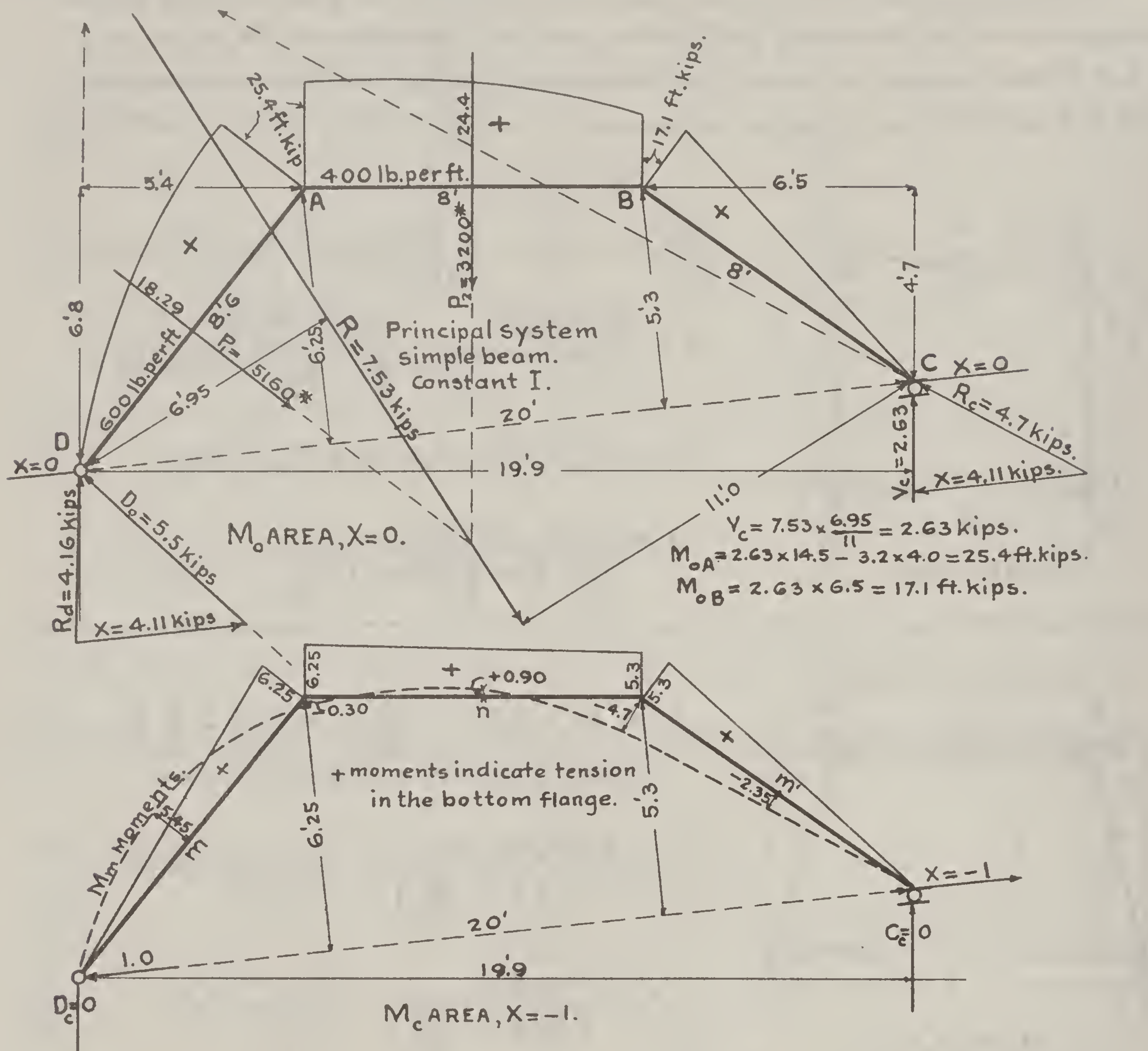
M_c AREA, X=-1.

$$\frac{1}{I}\int M_c^2 dx = \frac{2h}{3I}\times h^2 + \frac{2s}{3I_o}\left[h^2+hh_1+h_1^2\right].$$

$$X = \frac{\frac{1}{I}\int M_o M_c dx}{\frac{2h^3}{3I}+\frac{2s}{3I_o}\left[h^2+hh_1+h_1^2\right]}, \quad M_m = M_o - M_c X.$$

$$X_t = \frac{3\varepsilon t \ell E}{\frac{2h^3}{I}+\frac{2s}{I_o}\left(h^2+hh_1+h_1^2\right)}, \quad M_{mt} = -M_c X_t.$$

PROBLEM 29-U. TWO-HINGED, UNSYMMETRIC FRAME.



$$\int M_o M_c\,dx = \overbrace{\frac{8.6\times25.4}{4}\times\frac{5\times6.25}{3}}^{\text{Case 10}} + \overbrace{\frac{8}{24}\Big[5\big(6.25\times25.4-5.3\times17.1\big)+6\times24.4\big(6.25+5.3\big)+6.25\times17.1+5.3\times25.4\Big]}^{\text{Case 13}}$$

$$+\overbrace{\frac{8}{6}\times5.3\times2\times17.1}^{\text{Case 3}} = 1{,}871$$

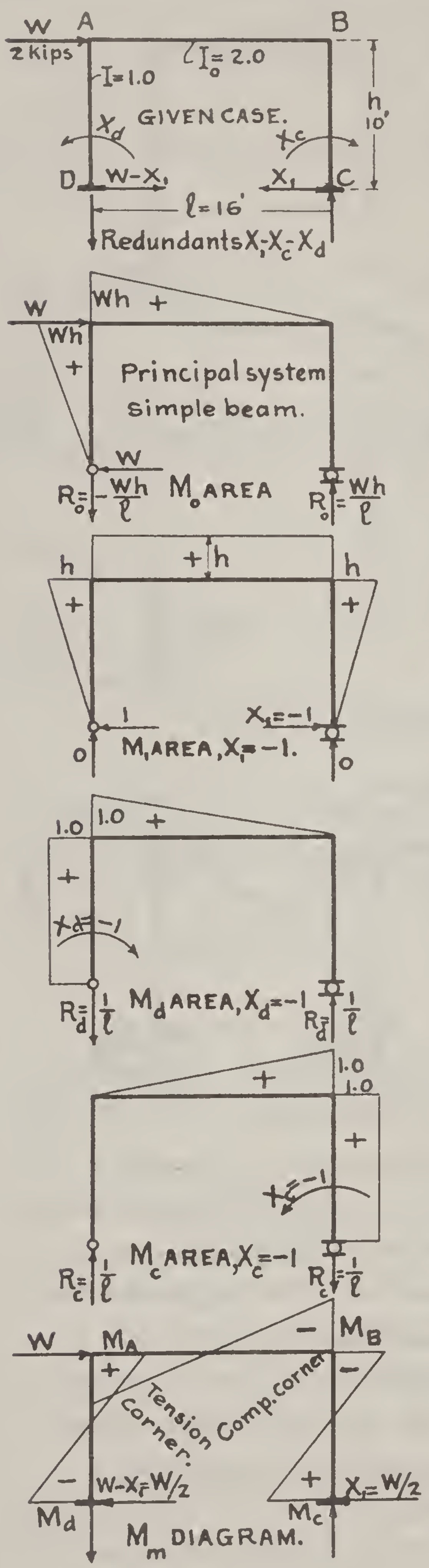
$$\int M_c^2\,dx = \overbrace{\frac{8.6}{3}\times\overline{6.25}^2}^{\text{Case 2}} + \overbrace{\frac{8}{3}\Big[\overline{6.25}^2+6.25\times5.3+\overline{5.3}^2\Big]}^{\text{Case 3}} + \overbrace{\frac{8}{3}\times\overline{5.3}^2}^{\text{Case 2}} = 456. \quad X=\frac{\int M_o M_c\,dx}{\int M_c^2\,dx}=\frac{1871}{456}=4.11\text{ kips.}$$

$M_m = M_o - M_c X = 18.29 - 4.11\times3.125 = +5.45$, $M_n = 24.4 - 4.11\times5.725 = +0.90$ ft. kips.

$M'_m = 8.55 - 4.11\times2.65 = -2.35$, $M_A = 25.4 - 4.11\times6.25 = -0.30$, $M_B = 17.1 - 4.11\times5.3 = -4.7$

<u>Reactions</u>. The loads on the span are first combined into a resultant $R = 7.53$ kips and the vertical reaction at C is computed as $V_c = 7.53\times\frac{6.95}{19.9} = 2.63$ kips. The reaction D_o is then found from the condition that R, V_c and D_o must intersect in a point, giving its lever arm as 15.1 ft. and making $D_o = \frac{7.53\times11}{15.1} = 5.5$ kips. R_d is finally found as the resultant of D_o and X, and R_c is the resultant of V_c and X. R_d and R_c are found graphically after X is computed as above.

PROBLEM 29-V. RIGID RECTANGULAR FRAME. THREE REDUNDANTS.



The general Eqs. 19-C applied to the present problem:

$$\frac{X_1}{I}\int M_1^2 dx + \frac{X_d}{I}\int M_1 M_d dx + \frac{X_c}{I}\int M_1 M_c dx = \frac{1}{I}\int M_o M_1 dx$$

$$\frac{X_1}{I}\int M_d M_1 dx + \frac{X_d}{I}\int M_d^2 dx + \frac{X_c}{I}\int M_d M_c dx = \frac{1}{I}\int M_o M_d dx$$

$$\frac{X_1}{I}\int M_c M_1 dx + \frac{X_d}{I}\int M_c M_d dx + \frac{X_c}{I}\int M_c^2 dx = \frac{1}{I}\int M_o M_c dx$$

$$\frac{1}{I}\int M_1^2 dx = \frac{2h^3}{3I} + \frac{\ell h^2}{I_o} = 1467.$$

$$\frac{1}{I}\int M_d^2 dx = \frac{h}{I} + \frac{\ell}{3I_o} = 12.67$$

$$\frac{1}{I}\int M_c^2 dx = \frac{h}{I} + \frac{\ell}{3I_o} = 12.67$$

$$\frac{1}{I}\int M_1 M_d dx = \frac{3h^2}{6I} + \frac{3\ell h}{6I_o} = 90.$$

$$\frac{1}{I}\int M_1 M_c dx = \frac{3h^2}{6I} + \frac{3\ell h}{6I_o} = 90.$$

$$\frac{1}{I}\int M_d M_c dx = \frac{\ell}{6I_o} = 1.333$$

$$\frac{1}{I}\int M_o M_1 dx = \frac{2Wh^3}{6I} + \frac{3W\ell h^2}{6I_o} = 1467$$

$$\frac{1}{I}\int M_o M_d dx = \frac{3Wh^2}{6I} + \frac{2W\ell h}{6I_o} = 153.3$$

$$\frac{1}{I}\int M_o M_c dx = \frac{W\ell h}{6I_o} = 26.67$$

The numerical elasticity equations thus become:

$$1467\,X_1 + 90.0\,X_d + 90.0\,X_c = 1467$$
$$90\,X_1 + 12.67\,X_d + 1.333\,X_c = 153.3$$
$$90\,X_1 + 1.333\,X_d + 12.67\,X_c = 26.67$$

The solution gives $X_1 = 1.00$, $X_d = 5.58$ and $X_c = -5.58$

The moment at any point m is given by Eq. 19B as

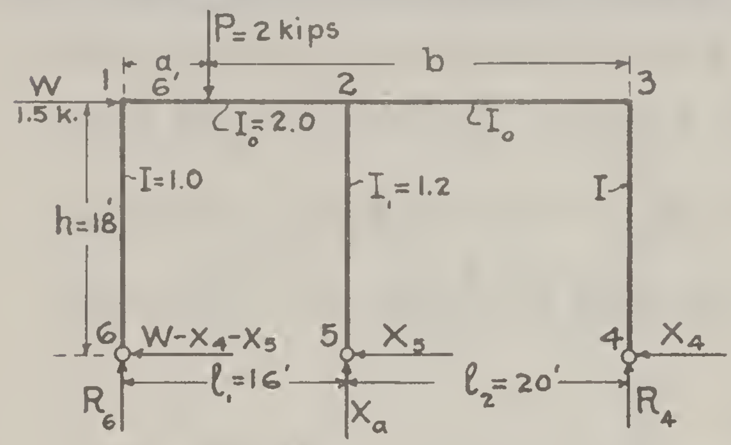
$$M_m = M_o - M_1X_1 - M_dX_d - M_cX_c$$

The four corner moments thus become:

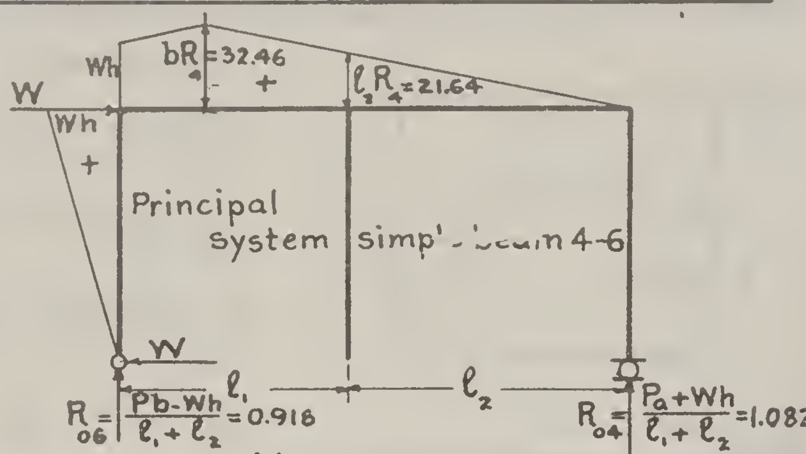
$M_A = Wh - h\times1 - 1\times5.58 + 0\times5.58 = +4.42$ ft. kips.
$M_B = 0 - h\times1 - 0 + 1\times5.58 = -4.42$ " "
$M_d = 0 - 0 - X_d - 0 = -5.58$ " "
$M_c = 0 - 0 - 0 + 1\times X_c = +5.58$ " "

<u>For different loads</u> on similar three sided rigid frames, only the three last integrals must be evaluated for each M_o area. As may be seen from Problem 29-U, any combination of simultaneous loads may be solved as one problem, merely by including all such loads in the M_o moment area.

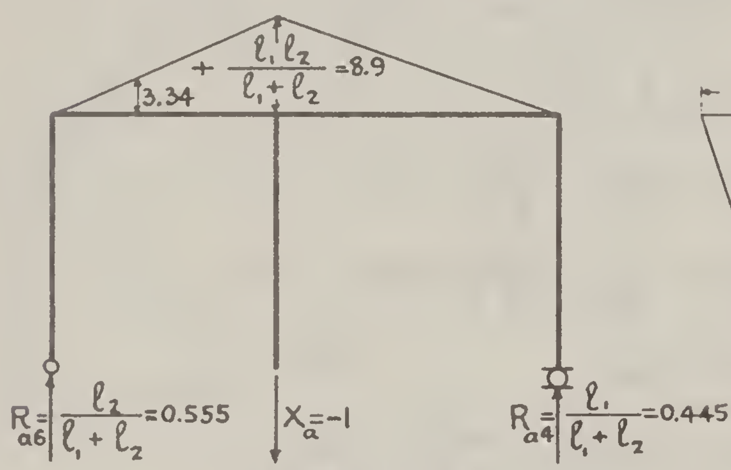
PROBLEM 29-W. DOUBLE RECTANGULAR FRAME, HINGED AT BOTTOM.



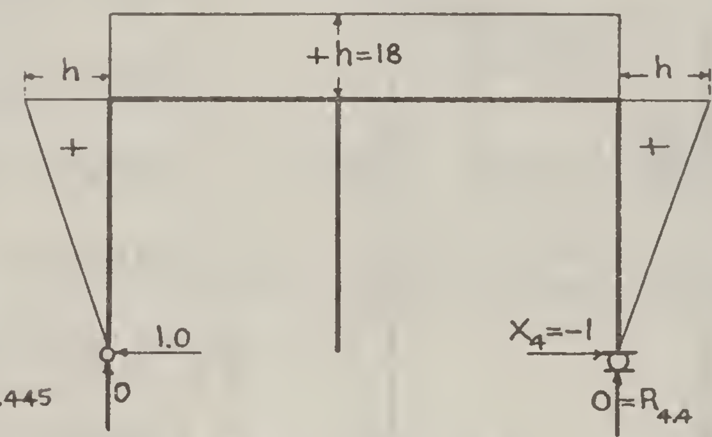
GIVEN CASE - 3 REDUNDANTS.



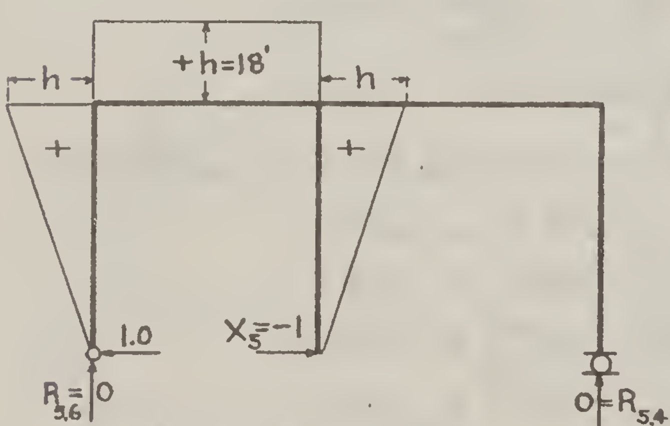
M_o AREA, ALL X's=O.



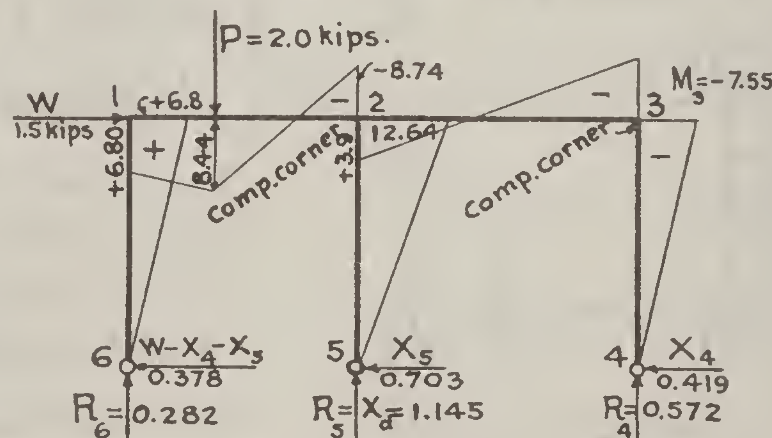
M_a AREA, $X_a=-1$.



M_4 AREA, $X_4=-1$.



M_5 AREA, $X_5=-1$



M_m MOMENTS.

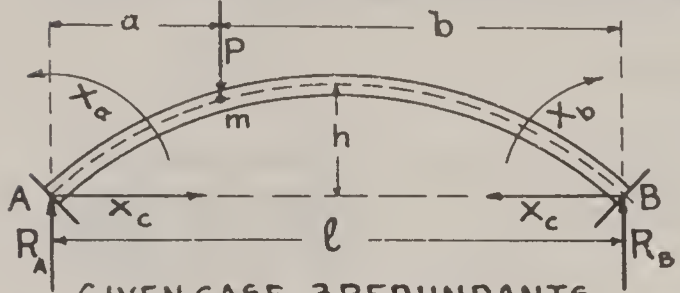
The complete solution of the above problem is suggested for practice.

The principal system, a simple beam 4-6, was selected as offering the easiest solution. Employing the numerical values, as given in the above sketch of the <u>given case</u>, the nine integrals for the solution of the three elasticity equations 19-C, may be evaluated by means of the substitution formulas Table 26-A. The numerical equations when solved, should give the following values: $X_a = 1.145$, $X_4 = 0.419$, and $X_5 = 0.703$ kips.

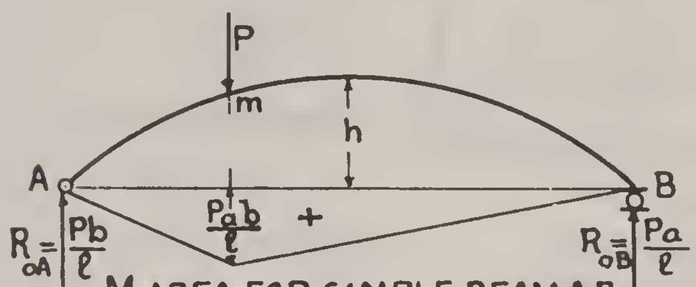
With these values of the X's find the moments at various points of the frame according to the general equation 19-B, thus- $M_m = M_o - M_a X_a - M_4 X_4 - M_5 X_5$, after which the M_m moment diagram may be drawn as shown.

<u>The reactions</u> are obtainable from $R = R_o - R_a X_a - R_4 X_4 - R_5 X_5$, noting that the vertical reactions of the M_4 and M_5 diagrams are all zero. The vertical reactions R_4, R_5 and R_6 depend on P, W and X and their sum must equal P. The sum of the horizontal reactions must equal W.

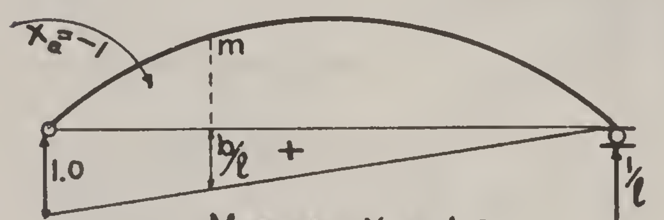
PROBLEM 29-X. FIXED PARABOLIC ARCH. CONSTANT EI.



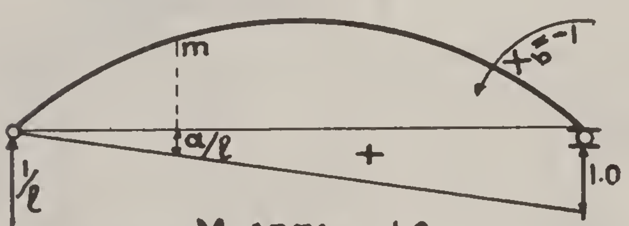
GIVEN CASE-3 REDUNDANTS.



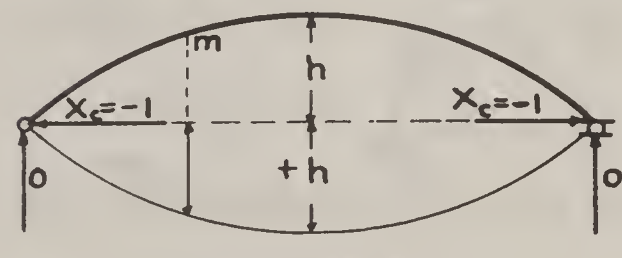
M_o AREA FOR SIMPLE BEAM A-B.



M_a AREA, $X_a=-1.0$



M_b AREA $=-1.0$



M_c AREA, $X_c=-1.0$

+ moments produce tension on the lower side of the arch.

Solution for a concentrated Load P.

The evaluation of 9 integrals by Table 26-A.

$\int M_a^2 dx = \ell/3$, $\int M_a M_b dx = \ell/6$, $\int M_o M_a dx = \frac{Pab}{6\ell}(\ell+b)$,

$\int M_b^2 dx = \ell/3$, $\int M_a M_c dx = \frac{\ell h}{3}$, $\int M_o M_b dx = \frac{Pab}{6\ell}(\ell+a)$,

$\int M_c^2 dx = \frac{8}{15}\ell h^2$, $\int M_b M_c dx = \frac{\ell h}{3}$, $\int M_o M_c dx = \frac{Pabh}{3\ell}\left(\ell+\frac{ab}{\ell}\right)$.

These values substituted into Eqs. 19-C, give the following three elasticity equations for Constant EI:

$$\frac{\ell}{3}X_a + \frac{\ell}{6}X_b + \frac{\ell h}{3}X_c = \frac{Pab}{6}\left(1+\frac{b}{\ell}\right).$$
$$\frac{\ell}{6}X_a + \frac{\ell}{3}X_b + \frac{\ell h}{3}X_c = \frac{Pab}{6}\left(1+\frac{a}{\ell}\right).$$
$$\frac{\ell h}{3}X_a + \frac{\ell h}{3}X_b + \frac{8}{15}\ell h^2 X_c = \frac{Pabh}{3}\left(1+\frac{ab}{\ell^2}\right).$$

These equations are simplified to obtain:

$$\left.\begin{aligned} 2X_a + X_b + 2hX_c &= \frac{Pab}{\ell}\left(1+\frac{b}{\ell}\right) \\ X_a + 2X_b + 2hX_c &= \frac{Pab}{\ell}\left(1+\frac{a}{\ell}\right) \\ X_a + X_b + \frac{8}{5}hX_c &= \frac{Pab}{\ell}\left(1+\frac{ab}{\ell^2}\right) \end{aligned}\right\} \text{-------}(1)$$

The solution of Eqs.(1) gives:

$$\left.\begin{aligned} X_c &= \frac{15Pa^2b^2}{4h\ell^3} \\ X_b &= \frac{Pa^2b}{\ell^2}\left(1-2.5\frac{b}{\ell}\right) \\ X_a &= \frac{Pab^2}{\ell^2}\left(1-2.5\frac{a}{\ell}\right) \end{aligned}\right\} \text{----------------}(2)$$

For several loads P_1, P_2 etc., calculate the values in Eqs.(2) for each load separately and sum the results respectively for X_c, X_b, and X_a.

Finally from Eq. 19B, obtain the moment at any point m of the arch axis as:

$$M_m = M_o - M_aX_a - M_bX_b - M_cX_c \text{ ------------}(3)$$

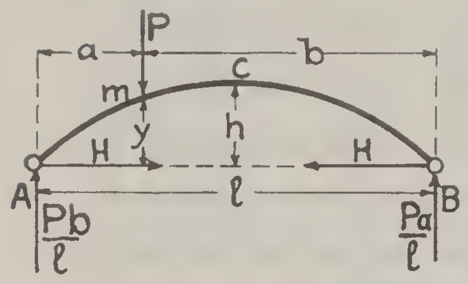
where M_o, M_a, M_b and M_c are the ordinates in the respective moment areas for the point m, which may be any axial point, not necessarily the loaded point.

For a uniform load p per ft. over the whole span, the redundants become $X_c = \frac{p\ell^2}{8h}$, and $X_b = X_a = 0$, giving a zero moment for all axial points, and thus proving that a parabolic arch is most economical.

The above formulas neglect axial thrust, as may be seen by referring to Eq. 14-E, and this affects X_c when the arch becomes very flat. See "Kinetic Theory of Engineering Structures" for full development of methods for bridges with moving loads.

FORMULAS FOR PARABOLIC AND SEMI-CIRCULAR ARCHES.

TWO-HINGED ARCHES.



Equation of parabola origin at A.

$$y = 4ab\frac{h}{\ell^2}.$$

$$H = \frac{5}{8}\frac{Pab}{\ell h}\left(1 + \frac{ab}{\ell^2}\right)$$

$$M_m = \frac{Pab}{\ell} - Hy.$$

$$M_c = \frac{Pa}{2} - Hh.$$

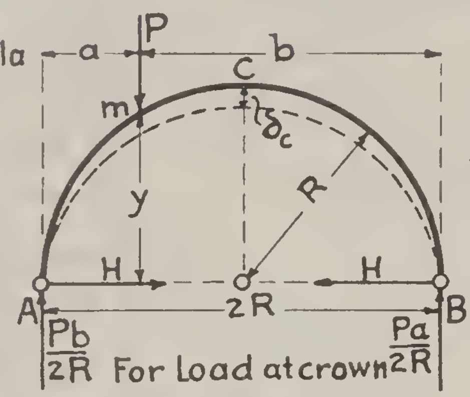
with load P at m.

see Problem 29M.

For load P at crown

$$H = 0.195\frac{P\ell}{h}$$

$$M_c = 0.055\,P\ell.$$



Equation of circle origin at center

$$y = \sqrt{R^2 - (R-a)^2} = a\sqrt{\frac{2R}{a} - 1}$$

$$H = \frac{Pab}{\pi R^2}$$

$$M_m = \frac{Pab}{2R} - Hy$$

$$M_c = \frac{Pa}{2} - HR$$

with load P at m.

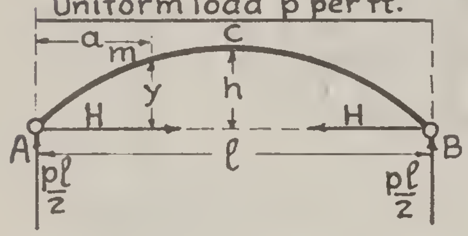
For Load at crown

$$H = 0.318\,P$$

$$M_c = 0.182\,PR$$

$$\delta_c = 0.019\frac{PR^3}{EI}.$$

Uniform load p per ft.



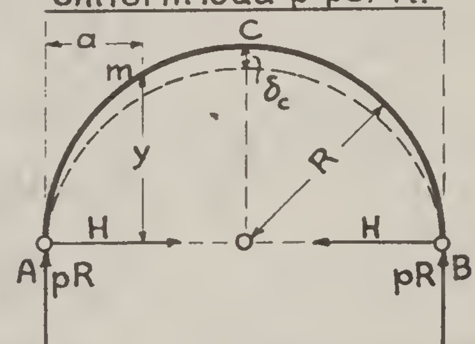
$$H = \frac{p\ell^2}{8h}$$

$$M_m = \frac{p\ell a}{2} - \frac{pa^2}{2} - Hy$$

$$= \frac{pab}{2} - Hy = 0$$

$$M_c = \frac{p\ell^2}{8} - Hh = 0$$

Uniform load p per ft.



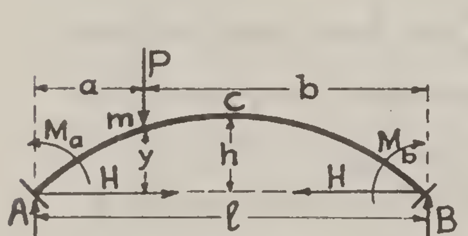
$$H = 0.4418\,pR$$

$$M_m = pRa - \frac{pa^2}{2} - Hy$$

$$M_c = 0.0582\,pR^2$$

$$\delta_c = 0.006\frac{pR^4}{EI}.$$

FIXED END ARCHES.



For load P at crown

$$H = \frac{15}{64} \times \frac{P\ell}{h}$$

$$M_a = M_b = -\frac{1}{32}P\ell.$$

$$M_c = +\frac{3}{64}P\ell.$$

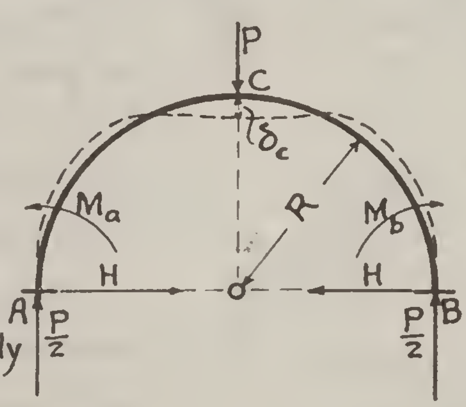
$$H = \frac{15Pa^2b^2}{4h\ell^3}$$

$$M_a = \frac{Pab^2}{\ell^2}\left(1 - 2.5\frac{a}{\ell}\right)$$

$$M_b = \frac{Pa^2b}{\ell^2}\left(1 - 2.5\frac{b}{\ell}\right)$$

$$M_m = \frac{Pab}{\ell} - M_a\frac{b}{\ell} - M_b\frac{a}{\ell} - Hy$$

see Problem 29X



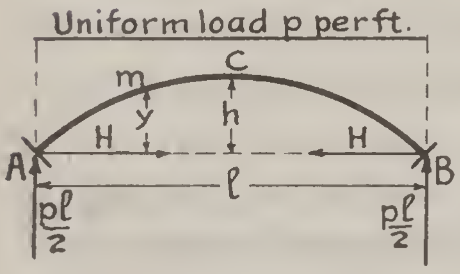
$$H = 0.470\,P$$

$$M_a = M_b = -0.118\,PR$$

$$M_c = +0.148\,PR$$

$$\delta_c = 0.0087\frac{PR^3}{EI}.$$

Uniform load p per ft.



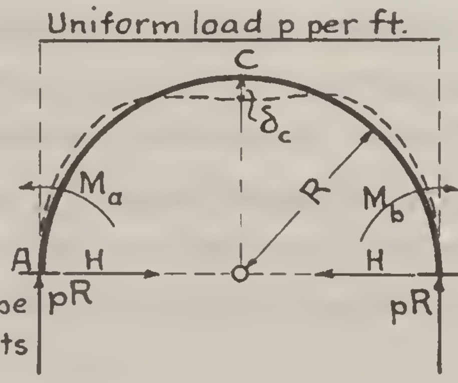
$$H = \frac{p\ell^2}{8h}$$

$$M_a = M_b = 0$$

$$M_m = M_{om} - Hy = 0$$

The parabolic axis is the most economical shape for uniform loads, because $M_m = 0$ for all points on the arch axis.

Uniform load p per ft.



$$H = 0.56\,pR$$

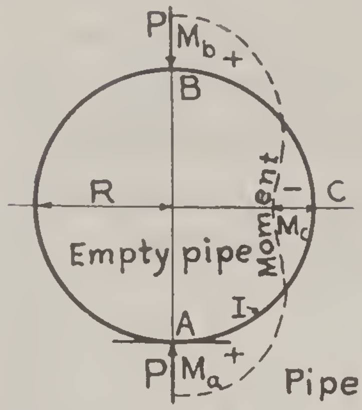
$$M_a = M_b = -0.106\,pR^2$$

$$M_c = 0.046\,pR^2$$

$$\delta_c = 0.005\frac{pR^4}{EI}$$

FORMULAS FOR CIRCULAR PIPES.

Loads in lbs. per inch of pipe. Dimensions inches. Positive moments - tension on inside.



$M_a = M_b = +0.318\,PR$

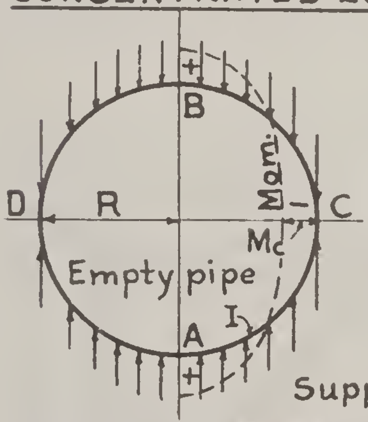
$M_c = -0.182\,PR$

Hor. dia. $= 2R + 0.14\,\frac{PR^3}{EI}$

Vert. dia. $= 2R - 0.149\,\frac{PR^3}{EI}$

Pipe supported at A.

CONCENTRATED LOAD ON TOP.



$M_a = M_b = +\frac{1}{4}\,g\,R^2$.

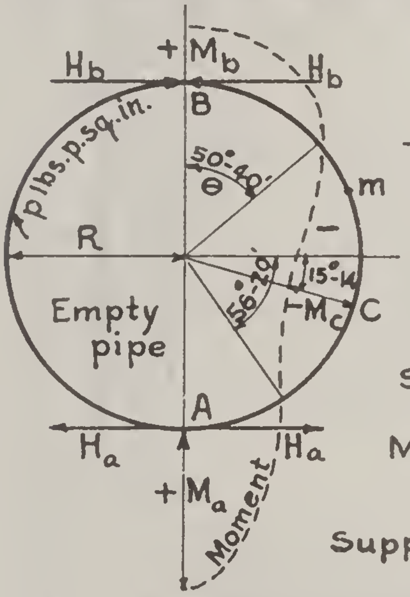
$M_c = -\frac{1}{4}\,g\,R^2$

Hor. dia. $= 2R + \frac{gR^4}{6EI}$

Vert. dia. $= 2R - \frac{gR^4}{6EI}$

Supported on DAC.

UNIFORM LOAD g LBS per HOR. INCH.



$-H_a = H_b = \frac{1}{2}pR$.

$M_a = 1.5\,pR^2$

$M_b = 0.5\,pR^2$

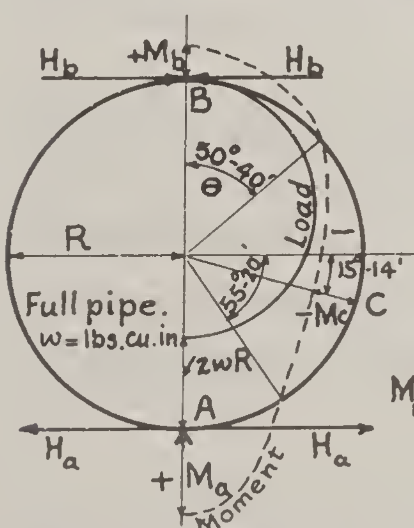
$M_c = -0.64\,pR^2$

Support A $= 2p\pi R$

$M_m = pR^2\left(1 - \frac{\cos\theta}{2} - \theta\sin\theta\right)$

Supported at A.

LOAD DEAD WEIGHT OF PIPE, p lbs. sq. in.



$H_a = \frac{5}{4}wR^2$, $H_b = \frac{3}{4}wR^2$

$M_a = 0.75\,wR^3$

$M_b = 0.25\,wR^3$

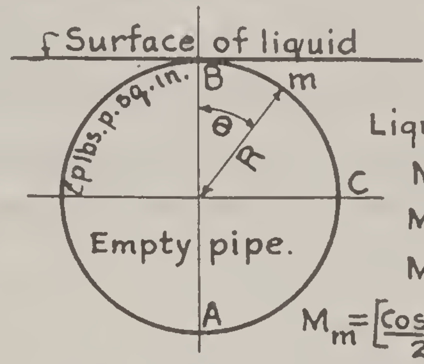
$M_c = -0.32\,wR^3$

Support A $= w\pi R^2$.

$M_m = \frac{1}{2}wR^3\left(1 - \frac{\cos\theta}{2} - \theta\sin\theta\right)$

$w = 0.0361$ lbs. cu. in. for water

PIPE FLOWING FULL - SUPPORTED AT A.



Liquid w lbs. per cu. inch.

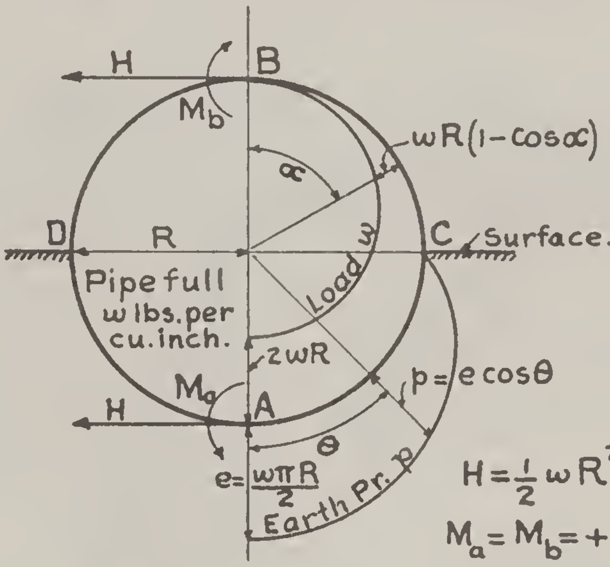
$M_a = -1.5\,R^2\left(\frac{1}{2}wR - p\right)$

$M_b = -0.5\,R^2\left(\frac{1}{2}wR - p\right)$

$M_c = +0.571\,R^2\left(\frac{1}{2}wR - p\right)$

$M_m = \left[\frac{\cos\theta}{2} + \theta\sin\theta - 1\right]\left(\frac{1}{2}wR - p\right)R^2$.

EMPTY FLOATING CYLINDER.



$H = \frac{1}{2}wR^2$

$M_a = M_b = +0.137\,wR^3$

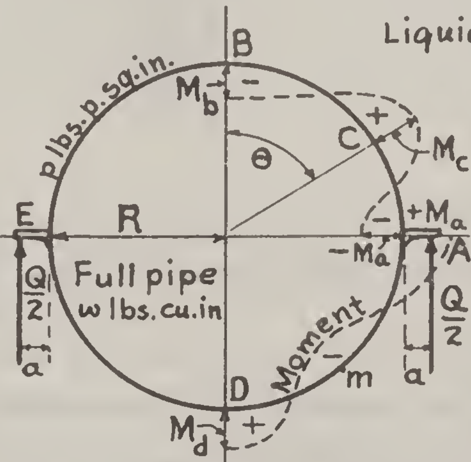
$M_c = -0.149\,wR^3$

PIPE FLOWING FULL - SUPPORTED ON EARTH.

Wgt. of pipe = p lbs. p. sq. in.

Liquid w lbs. p. cu. in.

Total weight = $Q = 2p\pi R + w\pi R^2$ lbs. per 1" of pipe.



$M_b = -M_d = \frac{QR}{\pi}\left(\frac{0.215\,a}{R} - 0.035\right)$.

$M_m = \frac{QR}{2\pi}\left[\theta\sin\theta + \frac{3}{2}\cos\theta - \frac{\pi}{2} + \frac{2a}{R}\left(\cos\theta - \frac{\pi}{4}\right)\right]$ bet. A and B.

$M_m = -\frac{QR}{2\pi}\left[(\pi - \theta)\sin\theta - \frac{3}{2}\cos\theta - \frac{\pi}{2} - \frac{2a}{R}\left(\cos\theta + \frac{\pi}{4}\right)\right]$ A to D.

$M_a = \pm\, Q\frac{a}{4}$. Max M_c when $\theta\tan\theta = \frac{1}{2} + \frac{2a}{R}$.

When $\frac{a}{R} = 0.04$, then $M_b = -M_d = -0.0085\,QR$,

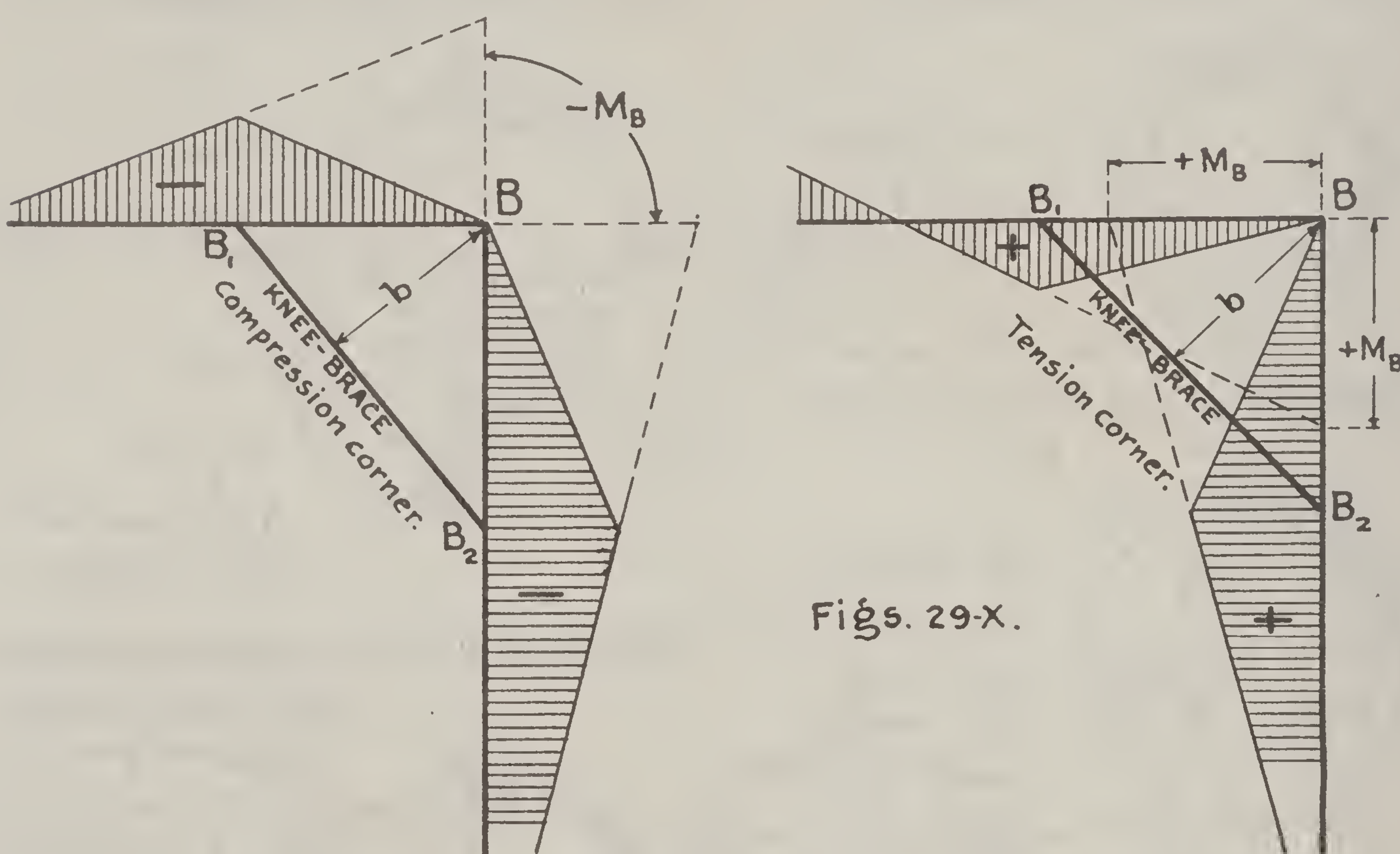
$M_a = \pm\,0.01\,QR$. max $M_c = 0.01\,QR$ for $\theta = 61°39'$

PIPE FLOWING FULL - SUPPORTED AT A and E.

PROBLEM 29-Y. STRESSES IN KNEE-BRACES.

Knee-braces may be introduced into a frame to replace rigid corner connections, thus stiffening the connections and reducing deflections.

The corner moments of the frame are determined as above shown, after which the knee-braces are introducea and the stresses produced therein are derived from the corner moments as shown in the figures below.



Figs. 29-X.

The stress in knee-brace $B_1 B_2 = \frac{M_B}{b}$ $\begin{cases}\text{tension for positive moment.}\\ \text{compression for negative moment.}\end{cases}$

The modified corner moments are shown as shaded areas for the cases of positive and negative moments.

The sign convention employed in Problems 29A to 29X may be stated as follows: A positive moment acting on a beam, produces tension on the lower side. A positive moment acting on the corner of a rigid frame, Figs. 29-X, produces tension on the inside of the corner. The same rule applies to an interior column connection as at point 2, Problem 29-W, noting that the tension element is on the inside with the positive moment.

Chapter 7 - Wind Stresses in Building Frames

Art. 30. The Nature of the Problem. In Art. 29, a number of problems (29 F, 29 G, 29H and 29K) were solved which would indicate, to some extent, the innumerable obstacles to be overcome in analysing a large multiple story building frame. But before attempting an analysis of such a frame, a design must have been produced on the basis of certain assumptions so that interest is centered primarily on methods of rational and economic designing, rather than on methods of ascertaining the possible stresses induced by wind forces in a given design, which may have been based on erroneous assumptions.

The numerous assumptions necessary to bring this problem within the realm of possible solution are so grossly approximate that any attempted refinement must be excluded on the grounds of imparting fictitious accuracy to something which is nevertheless unknowable. But, since the fundamental requirements of economy and simplicity governing the design of building frames are quite well established, it would appear futile to design on any other lines. It would also be wasted energy to apply some so-called accurate, but laborious method of testing a poorly conceived design, merely to discover something which was knowable in advance.

The value and scope of application of all methods of analysis, based on the theory of elasticity as applied to a strictly engineering structure, is in no wise questioned. However, a steel building may be classed as an engineering structure only when the frame is bare; but when clothed with architectural coverings of concrete, stone and brick, with steel and concrete floors, tile partitions, etc., it becomes a composite structure, the nature of which cannot be appraised in terms of mathematics.

Art. 31.
The Validity of the Assumptions. In dealing with the bare steel frame, the fundamental assumptions on which are based the so-called exact methods of stress analysis, may be evaluated, approximately, as follows:-

1. The connections between columns and beams or girders, are perfectly rigid. This may influence the resulting stresses by 30 to 50% as indicated by strain-gage measurements on riveted connections which are more or less flexible.

2. The changes in lengths of members due to direct stress are negligible. This may involve errors of perhaps 1 percent.

3. The lengths of the members are the distances between intersections of their neutral axes. This may affect the stresses by from 5 to 10 percent.

4. The deflection of a member due to internal shear is negligible. Deflections in beams are usually about 10% greater when web shear is considered.

5. The wind load is resisted entirely by the steel frame. If this were true, tall buildings would sway in the wind three to four times as much as they actually do. In other words, the architectural clothing, and not the steel frame alone, is responsible for the rigidity found by experience to be manifested in present day tall buildings. Were it not for the inherent rigidity of the architectural clothing of the steel frame, this assumption would be seriously deficient. Fortunately, the combination of a flexible steel frame, designed for certain wind loads and unit stresses, with a comparatively rigid fill material like concrete, does produce tall buildings with tenantable rigidity, a fact which has been demonstrated by successive development in building construction to greater and greater heights.

By the method illustrated in Problem 29 K, the horizontal deflection was computed for a 40 story building bent, 100 ft. wide by 467 ft. high, with seven columns spaced nearly equi-distant. The frame was designed for 15 lb. per sq. ft. wind pressure and unit stress of 24,000 lbs per sq. in. in bending. The bent carried wind over a width of 27 ft. The deflection for the bare frame carrying the full wind load, was found to be 6.5 in.

If such a building, in its completed state, was to sway as much as 1 in. in the wind, it might border on financial failure from the standpoint of occupancy.

There is, however, one mitigating circumstance which may make these vibratory disturbances more tolerable, and that is the period of the vibration which may vary from 5 to 10 seconds.

An approximate idea is thus given, of the proportion of wind load carried by the steel frame and that carried by the fill material even though the steel frame was designed to carry the entire wind load.

However, this is the price to be paid to obtain the necessary rigidity for a tall building, as rigidity constitutes an important asset to the owner, and the amount which the steel frame alone contributes to this factor, despite the severe load assumption, is in fact relatively insufficient to supply the necessity.

The assumption that the steel frame resists the entire wind load has produced certain empirical results in modern buildings which must be carefully weighed before launching out on new lines of design or new assumptions as to wind loads and allowable unit stresses. Thus, silicon steel stressed to 30,000 lbs. per sq. in., would contribute 25% greater deflection than carbon steel at 24,000 lbs. unit stress.

In the face of these facts, what useful purpose can be served by accurate methods of stress analysis? They may appear pretentious enough; but designers are not justified in claiming for them any extraordinary degree of accuracy when applied to a tall building frame.

Art. 32. Stress Analysis of a Given Design. Since the design of any given building frame is necessarily accomplished by some approximate method based on more or less arbitrary assumptions, the stress analysis of such a design requires for its solution the application of the theory of elasticity, whether in the form of slope deflections, area moments, or work equations, from which the redundants may be evaluated. With the redundants determined from elasticity equations, the various moments and stresses become known. The points of contraflexure are thus fixed, and the analysis locates them as a consequence of the design.

If the design is based on certain preconceived rational assumptions, the resulting moments will naturally be consistent with those assumptions, but not otherwise.

A design based on a certain set of assumptions cannot be reviewed or checked by any approximate method based on other assumptions. Hence, the assumptions for any given design should always be stated, as otherwise no satisfactory numerical check is possible. Thus, suppose a building frame was designed by Fleming's Method No. 1, and in the absence of this information someone attempts to check the design by Fleming's Method No. 3. This would result in grossly misleading contradictions without revealing any actual errors.

A laboriously conducted analysis of a poorly conceived design would merely lead to the discovery that a redesign was necessary and the latter would have to be tested by a repetition of the same laborious analysis.

It is thus apparent that a rational method of approximate design is of the utmost importance, and that it is futile to attempt a so-called accurate analysis of a tall building frame because, in practice the result is probably no more trustworthy than that obtainable by a simple, rational, approximate method, such as given in Art. 33.

Art. 33. Approximate Method of Design. With regard to methods of design, the objective aimed at, in terms of economy and constructive simplicity, can be clearly stipulated. Certainly, such aims can and should be embodied in the design with advantage over every other disposition that may be made. The only feasible method of design is one based on rational assumptions, economic principles, and enduring satisfaction as proved by actual results in practice.

The important stresses in a building frame are as follows:

1. For columns, direct axial and bending stresses due to lateral wind pressure, combined with dead and live load axial stresses.

2. For beams, bending stresses due to lateral wind pressure alone or combined with those due to dead and live floor loads. Also end shears on beams, due to wind alone, or combined with those due to dead and live floor loads.

If the points of contraflexure in the beams and columns can be approximately located, then all the critical wind stresses in any given frame are readily found for purposes of design without resorting to methods based on the theory of elasticity. The theory of elasticity is used essentially to locate these points of contraflexure, but if their location is already approximately known, then this laborious operation can be dispensed with.

The maximum economy in design for columns and beams, with respect to bending stresses, is attained when the points of contraflexure in the columns are at mid-story heights, and the points of contraflexure in the beams are at mid-points of spans. With respect to direct axial stresses in the columns, the maximum economy obtains when all or most of the direct stress is carried by the outside columns, and little or none by the interior columns, a condition which may be attained by an appropriate division of the total wind shear among the several columns at any floor.

Investigations made along these lines, prove these statements to be true for bents of equal spans and for all except the top and bottom stories of a frame. The top story presents no wind problem and the bottom story can be separately treated if greater accuracy is desirable by a method given in Art. 34.

When the spans of a bent are unequal, a further condition must be imposed, if simplicity in design is to be preserved. This is accomplished by assuming the bending moments at the ends of all beams in the same floor level as equal, allowing the beam shears to vary inversely as the span lengths.

The solution of any frame is now definitely fixed so long as the foregoing principles are not violated. The following steps will solve the problem for any given floor level, as illustrated on Fig. 33 A and described as follows:

1. Divide the total wind shear acting at any story, so that the two outside columns get equal amounts, one-half as great as for the interior columns. Then for n columns and a total horizontal shear W, each outside column will get a shear $H = \frac{W}{2(n-1)}$, and each interior column will get a shear of $2H$.

2. The beam shear in the outside span is then found from the shears in the outside column immediately above and below the beam in question, according to the condition that the beam moment must equal the sum of the two column moments. Since all beam moments at the same floor level are to be equal, then for unequal spans the beam shears will vary inversely as these span lengths.

Fig. 33 A and the accompanying data give a complete outline of this method based on the assumptions first published by Albert Smith, M. am. Soc. C. E. in the Journal of the Western Soc. of Engrs. Vol. XX, No. 4, P. 341, 1915.

The assumptions on which this approximate method is based, are not seriously in error except in the bottom or first story, and in this case, the only modification that might be desirable would be to select a more appropriate point of contraflexure in the first story, outside column, retaining all other assumptions as for the stories above. Thus, the moment at the base of the outside first story column might be made equal to $m = 0.7hH$ instead of $0.5hH$.

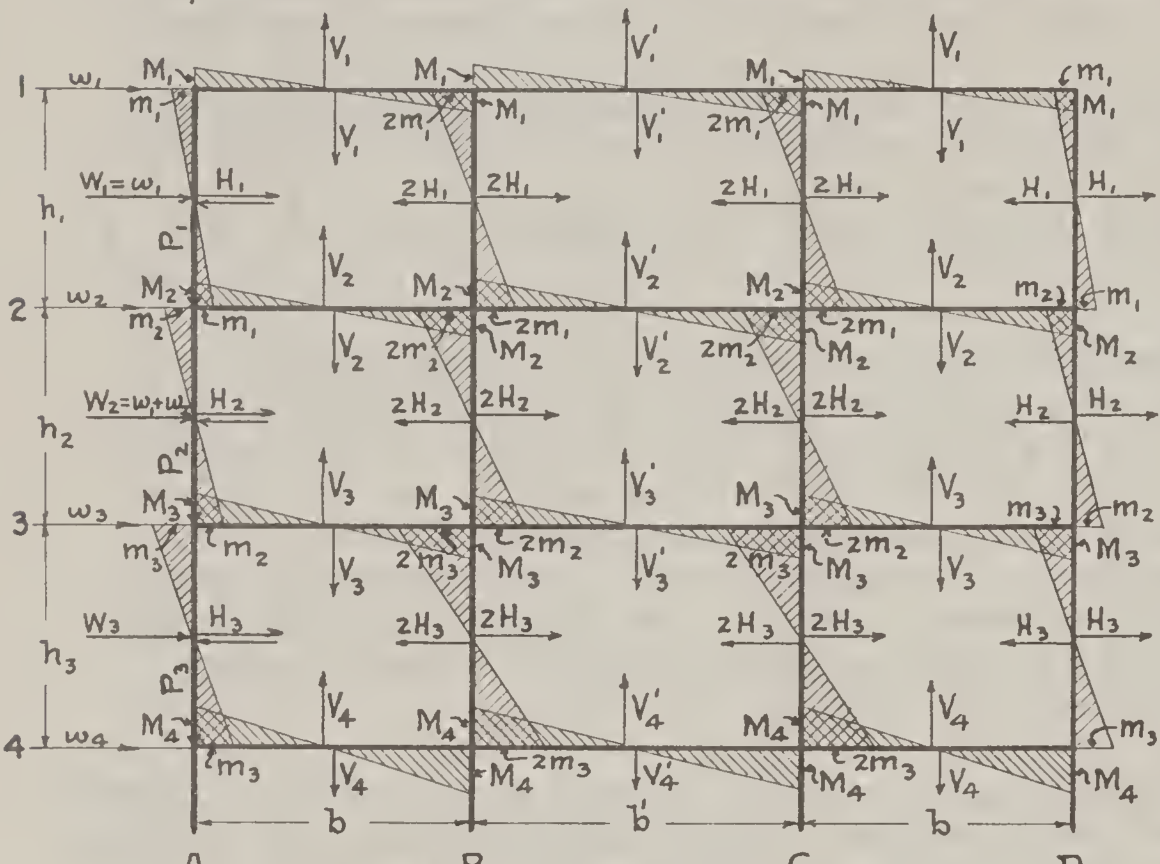
The approximate method of design by which points of contraflexure are assumed, yield moments at the joints which may be developed by appropriate details for connections, and the author maintains that such a structure must function in accordance with the disposition of metal provided in the design. There is nothing seriously in error with this procedure, and it is the only method available.

In the event that the structure is capable of offering sufficient internal work of resistance (without being stressed beyond the allowable limits) to balance the externally applied work, then the design must be regarded as adequate. This statement remains true even if certain parts are overstressed, provided that certain other parts are sufficiently understressed, in which case a readjustment takes place between the several redundants and the principal members, resulting in a slight permanent deformation of the overstressed connections. After such readjustment is once established by a single instance of maximum over-stress in certain parts, the structure will function normally thereafter without again overstressing these same parts.

WIND STRESSES IN BUILDING FRAMES.

ASSUMPTIONS.

1. Columns have points of contraflexure at mid-story heights.
2. Beams have points of contraflexure at mid spans.
3. Wind loads are divided so that interior columns take equal shears of twice the amount taken by each exterior column. All beam moments of same floor are equal.



DEFINITIONS.

w = wind load at a floor.

W = total wind load from roof down to a certain story.

H = hor. wind shear on an ext. column at mid-story height.

V = vert. wind shear at center of any beam.

P = axial wind load on an exterior column.

h = story height.

b = span of a beam.

n = number of cols. in bent.

All loads and forces in kips, All dimensions in feet. Column moments m, and beam moments M in kipft. Number floors from roof down.

FIG. 33A. SECTION OF ONE BENT.

Tabulate computations in following order, writing values on an outline diagram of the frame.

1. Find wind loads w_1, w_2, w_3, etc. acting at roof and the several floors.
2. Find total wind loads $W_1=w_1$; $W_2=w_1+w_2$; $W_3=w_1+w_2+w_3$; $W_m=\Sigma_1^m w$.
3. Find hor. wind shears H in one ext. column. $H_1=\frac{W_1}{2(n-1)}$; $H_2=\frac{W_2}{2(n-1)}$; $H_3=\frac{W_3}{2(n-1)}$; etc.
4. Find vert. shears V in outside span b as $V_1=\frac{H_1h_1}{b}$; $V_2=\frac{1}{b}[H_1h_1+H_2h_2]$; $V_3=\frac{1}{b}[H_2h_2+H_3h_3]$; etc.
5. Find beam moments - $M_1=V_1\frac{b}{2}$; $M_2=V_2\frac{b}{2}$; $M_3=V_3\frac{b}{2}$; etc. equal for all beams of same floor.
6. Find outside column moments, $m_1=\frac{H_1h_1}{2}$; $m_2=\frac{H_2h_2}{2}$; $m_3=\frac{H_3h_3}{2}$; etc.
7. Make inside column moments double those of outside columns.
8. Find outside column loads $P_1=V_1$; $P_2=V_1+V_2$; $P_3=V_1+V_2+V_3$; $P_n=\Sigma_1^n V$.
9. For equal bays, inside columns get no direct load. For unequal bays, inside columns get direct load $P_n=\Sigma_1^n V-\Sigma_1^n V'$, where $V'=\frac{b}{b'}V$.

Note: In preparing a wind stress diagram, the following items only are given on the diagram:

1. Beam moment for each floor level, constant for same floor.
2. Moments in exterior columns. The moments in interior columns are twice as great.
3. Direct axial loads in exterior columns, also in interior columns when bays are unequal.

D. Molitor, C.E.
Aug. 1926.

Art. 34. More Accurate Solution for Bottom Stories of a Frame in Conjunction with an Approximate Solution of the Upper Stories.

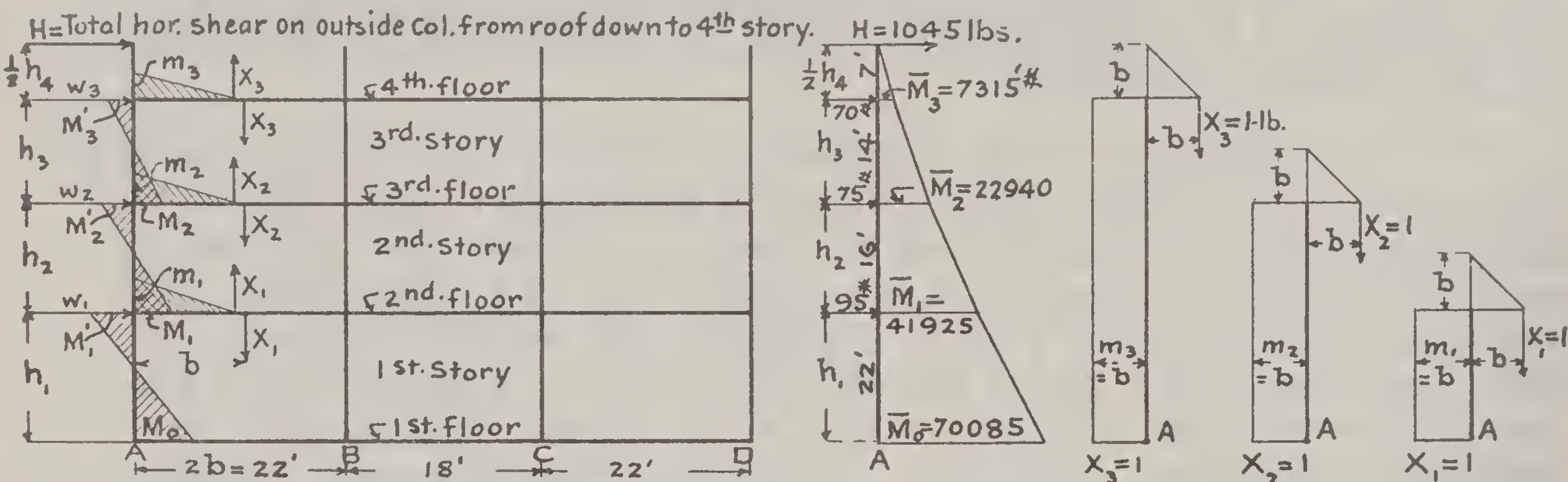
Should it be desired to solve the first three stories of a frame with greater accuracy than that afforded by the approximate design, then the following method, illustrated in Fig. 34-A, may be employed. It does not depend on assumed points of contraflexure of the columns, but does embody the other assumptions as to distribution of the total wind shear among the several columns, and that the beam moments are equal in the same floor level as in Fig. 33-A.

The method is the same as given in Problem 29-H, except that the effect of the upper stories above the fourth floor is represented merely by the total wind shear from the roof down to the point of contraflexure of the fourth story outside column A. This is possible because the sum of the moments of all the wind shears ω and beam shears X, acting on the exterior columns, must be zero about any point of contraflexure in the column.

Hence in Fig. 34-A, the total horizontal shear H, on the outside column, due to wind from the roof down to the fourth story, is the resultant external force acting above the fourth floor and must be considered in producing the M_o moment area.

The error in assuming H to act at $\frac{1}{2}h_4$ above the fourth floor, is relatively small and becomes negligible for the lower stories.

MORE ACCURATE SOLUTION FOR THREE BOTTOM STORIES OF A FRAME.
IN CONJUNCTION WITH APPROXIMATE SOLUTION OF UPPER STORIES.



Four bottom stories of given frame. FIG. 34A. $\overline{M}_0$ area. m_3 area. m_2 area. m_1 area.

The wind loads $w_1, w_2; w_3$, and the hor. shear H, are those acting on Column A only.

The ordinates of the $\overline{M}_0$ area are the cantilever moments of the wind loads w_1, w_2 and w_3, and the shear H about the 1st., 2nd, 3rd and 4th floor levels, respectively, assuming the frame cut by a vertical section through the points of contraflexure of the beams in the 1st span.

All external forces act on column A as a principal system with moments as follows:

$$\left.\begin{array}{l} \overline{M}_0 = H\left[h_1+h_2+h_3+\frac{h_4}{2}\right] + w_3\left[h_1+h_2+h_3\right] + w_2\left[h_1+h_2\right] + w_1h_1 . \\ \overline{M}_1 = H\left[h_2+h_3+\frac{h_4}{2}\right] + w_3\left[h_2+h_3\right] + w_2h_2 \;;\; \overline{M}_2 = H\left[h_3+\frac{h_4}{2}\right] + w_3h_3 \;;\text{and } \overline{M}_3 = \frac{Hh_4}{2} \end{array}\right\} \text{---- Eq. 1.}$$

The ELASTICITY EQUATIONS for the 3 unknown shears X_1, X_2 and X_3, are:

$$\left.\begin{array}{llll} X_1\int m_1^2dx & + X_2\int m_1m_2\,dx & + X_3\int m_1m_3\,dx & = \int \overline{M}_0m_1\,dx \\ X_1\int m_2m_1dx & + X_2\int m_2^2\,dx & + X_3\int m_2m_3\,dx & = \int \overline{M}_0m_2\,dx \\ X_1\int m_3m_1dx & + X_2\int m_3m_2dx & + X_3\int m_3^2\,dx & = \int \overline{M}_0m_3\,dx \end{array}\right\} \text{--------- Eq. 2.}$$

The integrals in these Eqs. 2, are evaluated once for all as follows:

$$\left.\begin{array}{ll} \int \overline{M}_0m_1\,dx = \frac{bh_1}{2}(\overline{M}_1+\overline{M}_0)\;; & \int m_1^2dx = b^2\left(h_1+\frac{b}{3}\right) \\ \int \overline{M}_0m_2\,dx = \frac{bh_2}{2}(\overline{M}_2+\overline{M}_1) + \frac{bh_1}{2}(\overline{M}_1+\overline{M}_0)\;; & \int m_2^2dx = b^2\left(h_1+h_2+\frac{b}{3}\right) \\ \int \overline{M}_0m_3dx = \frac{bh_3}{2}(\overline{M}_3+\overline{M}_2) + \frac{bh_2}{2}(\overline{M}_2+\overline{M}_1) + \frac{bh_1}{2}(\overline{M}_1+\overline{M}_0)\;; & \int m_3^2dx = b^2\left(h_1+h_2+h_3+\frac{b}{3}\right) \\ \int m_1m_2dx = h_1b^2 \;;\; \int m_1m_3\,dx = h_1b^2\;; \text{ and } \int m_2m_3dx = b^2(h_1+h_2). & \end{array}\right\} \text{--- Eq. 3.}$$

COLUMN MOMENTS. — BEAM MOMENTS.

$$\left.\begin{array}{lll} M_0 = \overline{M}_0 - b(X_3+X_2+X_1)\;; & M_2' = \overline{M}_2 - b(X_3+X_2). & \text{2nd Floor } m_1 = bX_1 \\ M_1' = \overline{M}_1 - b(X_3+X_2+X_1)\;; & M_2 = \overline{M}_2 - bX_3 . & \text{3rd Floor } m_2 = bX_2 \\ M_1 = \overline{M}_1 - b(X_3+X_2) \;; & M_3' = \overline{M}_3 - bX_3 . & \text{4th Floor } m_3 = bX_3 \end{array}\right\} \text{--- Eq. 4.}$$

PROBLEM: Fig. 5, p. 20 - Bull. 40, Univ. of Ill., Wilson & Maney. Total shear 4th story = 6270# or $H = \frac{6270}{2(n-1)} = 1045$#. The wind loads w_1, w_2, w_3 for Col. A only, are 1/6 the total wind load at the respective floors. Inserting these loads and the dimensions into the above $\overline{M}_0$ area, then the integral Eqs. 3 can be solved to obtain the following numerical elasticity Eqs. 2:

$$\left.\begin{array}{l} 3106\,X_1 + 2662\,X_2 + 2662\,X_3 = 13{,}553{,}210 \\ 2662\,X_1 + 5042\,X_2 + 4598\,X_3 = 19{,}261{,}240 \\ 2662\,X_1 + 4598\,X_2 + 6736\,X_3 = 21{,}590{,}880 \end{array}\right\} \text{ Solved with a 10" slide rule give: } \left\{\begin{array}{l} X_1 = 1790 \text{ lbs.} \\ X_2 = 1590 \text{ "} \\ X_3 = 1420 \text{ "} \end{array}\right.$$

Hence from Eqs. 4:

$M_0 = 70{,}085 - 11\times4800 = 17{,}285$ ft.lbs.	$M_2' = 22940 - 11\times3010 = -10{,}170$ ft.lbs.	$m_1 = 19{,}700$ ft.lbs.
$M_1' = 41{,}925 - 11\times4800 = -10{,}875$ " "	$M_2 = 22940 - 11\times1420 = 7{,}320$ " "	$m_2 = 17{,}500$ " "
$M_1 = 41{,}925 - 11\times3010 = 8{,}815$ " "	$M_3' = 7315 - 11\times1420 = -8{,}305$ " "	$m_3 = 15{,}620$ " "

Art. 35. Comparison of Results due to Several Methods. Table 35 A was compiled to illustrate relative merits of results obtainable by the four methods compared. The Slope-deflection analysis was taken from Bulletin 80, Eng. Experiment Station, Univ. of Illinois; the Ross figures are from a paper on "The Design of Tall Building Frames to resist Wind" by Ross and Morris in Proc. Am. Soc. C. E. May, 1928; and Fleming's method I was used in preference to the less rational Fleming Methods II and III.

It should be noted that in the Slope-deflection solution as well as in the Ross and Morris Method, the larger discrepancies, especially in the column moments, are due to the abrupt changes in moments of inertia of the columns, where the section just below a splice is always of excess area. In all approximate methods, where moments of inertia are not given much consideration, such effects are obscured, as well they might be.

A careful examination of ~~the~~ Table 35 A, will show that there is little, if any, choice between the tabulated results, hence, from the designer's standpoint, they are either equally accurate or equally erroneous.

The approximate method in Art. 33, is considered rational, and more practical than any of the other approximate methods, and being based primarily on economic considerations as well as simplicity, may well be accepted as standard practice.

TABLE 35A. COMPARISON OF RESULTS FROM VARIOUS METHODS.

20 STORY BUILDING, WILSON & MANEY, FIG. 5, BULLETIN 40, UNIV. OF ILL.

TABULATION OF MOMENTS FOR LOWER 12 STORIES FOR 30 LB. WIND ON 1 FT. WIDTH.

	TOTAL WIND LOAD W ON 1 FOOT WIDTH 30 lbs. sq. ft	MOMENTS COL. A FT. KIPS				MOMENTS COL. B. FT. KIPS.				GIRDER MOMENTS M_{AB} FT. KIPS.			
		SLOPE-DEFLECTION	ROSS-MORRIS	FLEMING I	MOLITOR	SLOPE-DEFLECTION	ROSS-MORRIS	FLEMING I	MOLITOR	SLOPE-DEFLECTION	ROSS-MORRIS.	FLEMING I	MOLITOR.
13	(13) STORY W = 2880 lbs. 9'6" FLOOR									6.12	6.10	6.02	6.12
	(12) 3240 12'	3.45	3.23			6.79	6.49						
				3.18	3.24			6.59	6.48				
12		2.84	3.23			6.26	6.50			6.82	6.81	6.69	6.84
	(11) 3600 12'	3.96	3.58			7.42	7.21						
				3.49	3.60			7.34	7.20				
11	Splice	3.46	3.55			7.60	7.26			7.47	7.43	7.26	7.56
	(10) 3960 12'	3.87	3.88			8.50	7.98						
				3.83	3.96			8.07	7.92				
10		3.53	3.86			7.85	8.01			7.92	8.06	8.07	8.28
	(9) 4320 12'	4.38	4.20			8.68	8.77						
				4.28	4.32			8.71	8.64				
9	Splice	4.32	4.22			8.61	8.73			7.98	8.80	9.06	9.00
	(8) 4680 12'	3.70	4.58			8.10	9.47						
				4.66	4.68			9.42	9.36				
8		5.64	4.69			10.62	9.35			10.49	9.74	9.54	9.84
	(7) 5040 12'	4.80	5.05			10.17	10.07						
				4.93	5.16			10.17	10.32				
7	Splice	5.00	5.07			10.33	10.05			11.25	11.46	11.10	11.49
	(6) 5430 14'	6.25	6.39			11.94	12.61						
				6.19	6.33			12.75	12.66				
6		6.93	6.53			12.97	12.49			13.97	13.56	12.90	13.15
	(5) 5850 14'	7.04	7.02			13.52	13.44						
				6.74	6.82			13.59	13.64				
5	Splice	6.90	6.82			13.50	13.63			14.21	14.14	14.10	14.13
	(4) 6270 14'	7.32	7.32			14.67	14.65						
				7.26	7.31			14.61	14.62				
4		7.32	7.52			14.46	14.43			15.59	15.54	14.85	15.11
	(3) 6690 14'	8.36	8.02			15.62	15.39						
				7.62	7.80			15.63	15.60				
3	Splice	7.56	7.80			15.17	15.62			16.92	17.30	16.65	17.30
	(2) 7140 16'	9.42	9.50			18.87	19.05						
				9.25	9.50			19.05	19.00				
2		8.96	10.28			19.92	18.30			23.98	25.33	23.27	23.63
	1ST STORY 7710 22'0"	14.90	13.88		14.13	20.92	21.79		28.30				
				13.91				28.60					
1	FLOOR	22.67	22.00		19.80[×]	25.68	27.14		28.30[†]	—	—	—	—

[×] 22'×0.7×1285* = 19800 Ft.lbs. [†] 22×0.5×1285×2 = 28300 Ft.lbs.

Art. 36. **Beam and Column Wind Connections.** Any beam subjected to floor loads and bending stresses due to lateral wind pressure, must be designed for the maximum combination of both. Thus for the dead and live loads alone on the basis of 18000 lbs. per sq. in. in bending and 12000 lbs. per sq. in. in shear, and for 24000 lbs. per sq. in. in bending and 15000 lbs. per sq. in. in shear for the combined wind and other load effects, according to the specifications of the Am. Inst. of Steel Constr.

Fig. 36 A illustrates this beam loading and the resulting moments and end shears governing the design of a beam and its connections with adjacent columns.

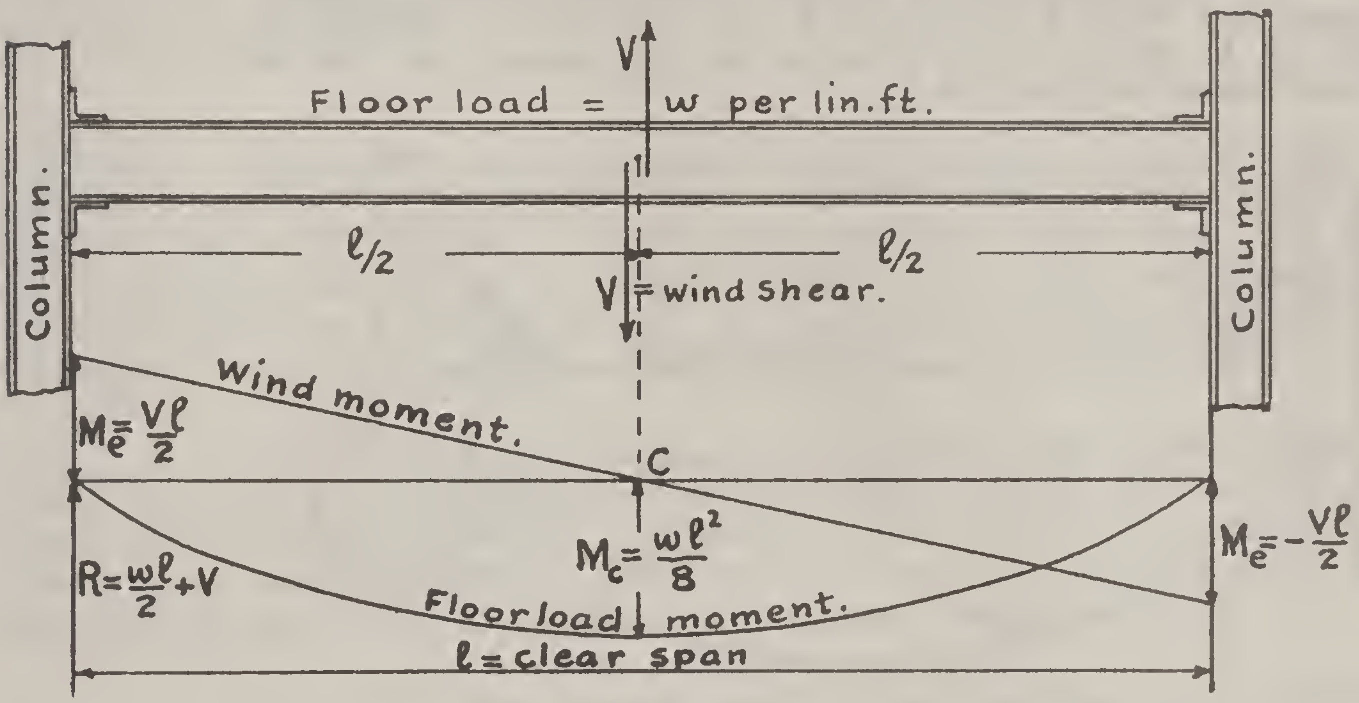


Fig. 36A.

$M_c = w\ell^2/8$ The maximum center moment due to floor load is for a simple beam without continuity, and the wind moment for the point C is zero.

The maximum wind moment occurs at the ends of the beam and is $\pm V\ell/2$, while the floor load moments at these ends are zero.

The maximum end shear is $R = \frac{w\ell}{2} + V$.

Hence the beam must be designed for the center moment M_c on the basis of 18000 lbs. per sq. in. unit stress in bending, and for the end moment M_e on the basis of 24000 lbs. per sq. in. unit stress in bending. The beam must, therefore, satisfy both moment requirements at center and ends, and since the end moment must be developed by a suitable end connection with the column, this latter condition, involving the strength of the end connection, will usually govern the size of the beam.

This emphasizes the importance of the end connections and unless these are so designed as to develop the full capacity of the beam in bending, then a beam of larger size must be selected for which a proper end connection becomes possible, and the beam itself will then be of excess size and be wasteful.

Also, since the end moment, due entirely to wind, usually governs the size of the beam and its end connections, therefore, it would be a mistake to increase the negative end wind moment by figuring continuity in the beam for floor load. In this case the combined positive end moment would be diminished, while the negative end moment would be increased.

For beam and Column Connections with knee braces the moment conditions are the same as shown in Fig. 36 A, while the stresses in the braced end are as illustrated in Fig. 36 B.

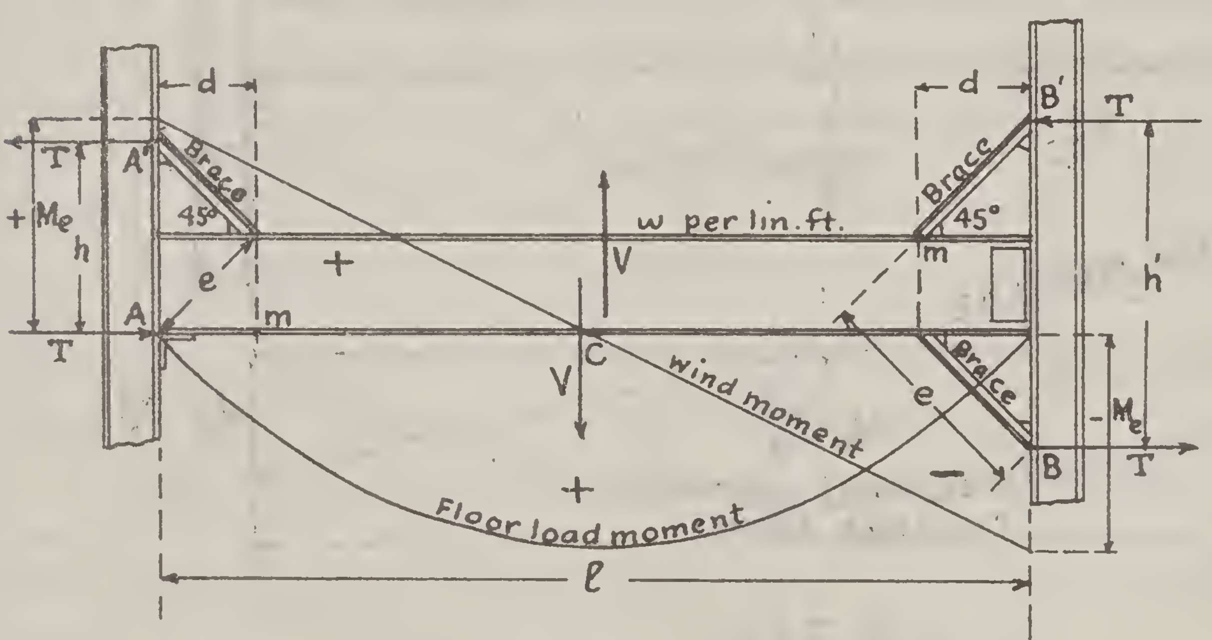


Fig. 36 B.

__For one brace as at A:__

Shear at column face $= \frac{w\ell}{2} \pm \frac{V}{d}\left(\frac{\ell}{2} - d\right)$ taken by beam web connection.

$M_e = \frac{V\ell}{2} = Th$, hence $T = \frac{V\ell}{2h}$ = tension or compression.

Stress in brace $= \pm\, T \sec. 45° = \frac{V\ell}{2h} \sec. 45° = \pm \frac{V\ell}{2e}$.

$M_m = V\left(\frac{\ell}{2} - d\right) + \frac{w}{2}\left(\ell d - d^2\right)$ and $M_c = \frac{w\ell^2}{8}$.

The points A and A' are the centers of rivet groups at the two connections with column A.

The bottom flange of the beam must be connected to the column at A for a tension T, and both end connections of the brace must be designed for a tension T and a shear T giving a direct stress in the brace of $\pm 1.41\,T$.

<u>For two braces as at B:</u>

Shear at col. face $= \frac{w\ell}{2} \pm \frac{V}{d}\left(\frac{\ell}{2} - d\right)$ taken by beam web connection.

$M_e = \frac{V\ell}{2} = Th'$, hence $T = \frac{V\ell}{2h'}$ = tension or compression.

Stress in brace $= \pm T \sec.45^\circ = \frac{V\ell}{2h'} \sec.45^\circ = \pm \frac{V\ell}{2e}$.

$M_m = V\left(\frac{\ell}{2} - d\right) + \frac{w}{2}\left(\ell d - d^2\right)$ and $M_c = \frac{w\ell^2}{8}$.

The points B and B' are the centers of rivet groups at the two connections. The stress in the braces is again $\pm 1.41\,T$.

Art. 37. <u>Details for Beam and Column Wind Connections.</u> In designing the beams and their connections, the procedure should be as follows: First design the beam for its floor load on the basis of a simple span, and then select an end connection from Table 37-A, sufficient to develop the required end wind moment for this beam.

In case the actual wind moment is greater than afforded by any end connection given in the table for a beam of this size, then the beam first selected must be increased to such size as the wind moment may require, using the strongest detail given for type T connections.

The smallest size beams, satisfying the center and end wind moment requirements, will result in the most economic design of the wind frame.

Table 37-A serves a very useful purpose and, except in extraordinary cases, will cover all standard connections in tall buildings.

DETAILS OF BEAM and COLUMN WIND CONNECTIONS.

TABLE 37A. MOMENTS IN FT. LBS. FOR STANDARD END CONNECTIONS OF BEAMS.

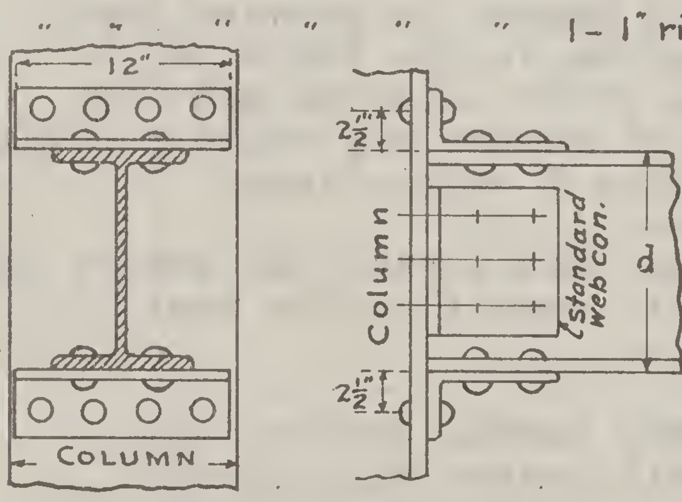
TYPE L CONNECTIONS				TYPE T CONNECTIONS.		
BEAM	2Ls 6×4×5/8-12" 3/4 RIVETS	2Ls 6×4×5/8-12" 7/8 RIVETS	2Ls 8×4×3/4-12" 1" RIVETS.	BEAM	1/2-27" I-101-lbs. 12" long-7/8" rivets	1/2-30" I-115-lbs. 14" long-1" rivets
12" CB25*	32,000	43,000	56,000	12" I 32*	72,000	—
14 " 30	37,000	50,000	65,000	14 I 33	95,000	—
15 " 37.3	40,000	53,000	70,000	15 I 42.9	108,000	118,000
16 " 35	42,000	57,000	75,000	16 I 38	115,000	118,000
18 " 47	48,000	65,000	84,000	18 I 47	130,000	169,000
20 BI56	53,000	72,000	93,000	20 I 56	143,000	188,000
21 CB58	56,000	75,000	98,000	21 I 58	151,000	197,000
22 BI58	58,000	79,000	102,000	22 I 58	158,000	207,000
24 CB70	64,000	86,000	112,000	24 I 70	172,000	226,000
27 " 91	72,000	97,000	126,000	27 I 91	194,000	254,000
30 " 115	80,000	107,000	140,000	30 I 115	216,000	282,000

Unit stress in bending, 24000 lbs. per sq. in. for wind moments.

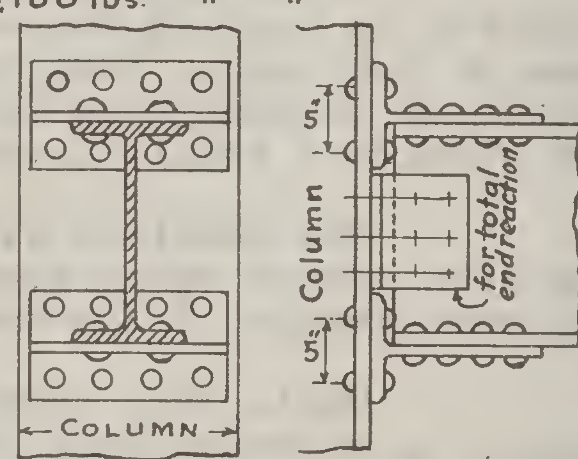
Stress in single shear or tension for 1-3/4" rivet = 7,950 lbs. for wind.

" " " " " " " 1-7/8" rivet = 10,800 lbs. " "

" " " " " " " 1-1" rivet = 14,100 lbs. " "



TYPE L CONNECTION.



TYPE T CONNECTION.

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Since the columns are usually 12" and 14" H columns from four to eight rivets can be utilized in the connection of each flange for type L connections, giving the following tension and shear values P, for a group of 4 rivets on the basis of 24,000 lbs. per sq. inch.

4 - 3/4" rivets	@ 7,950 lbs.		31,800 lbs.
4 - 7/8" "	@ 10,800 "		43,200 "
4 - 1" "	@ 14,100 "		56,400 "

Double these values are used for groups of 8 rivets in type T connections.

The moment of resistance of the end connection for a beam of any depth d is represented by the product Pd in foot lbs. as given in Table 37-A. The maximum value which Pd can have for any particular beam is $M = fS = \frac{24000}{12} S$, where S is the section modulus of the beam in inch units. It should be noted that the light weight sections are usually strong enough to develop a resisting moment equal to or greater than that of the best possible end connection, hence only light sections should be employed except when the floor load requires a heavier section.

There is one other point to be covered before concluding this subject, and that is the required thickness of metal in the connecting angles or tees, to develop the assumed rivet values P.

For type L connections.

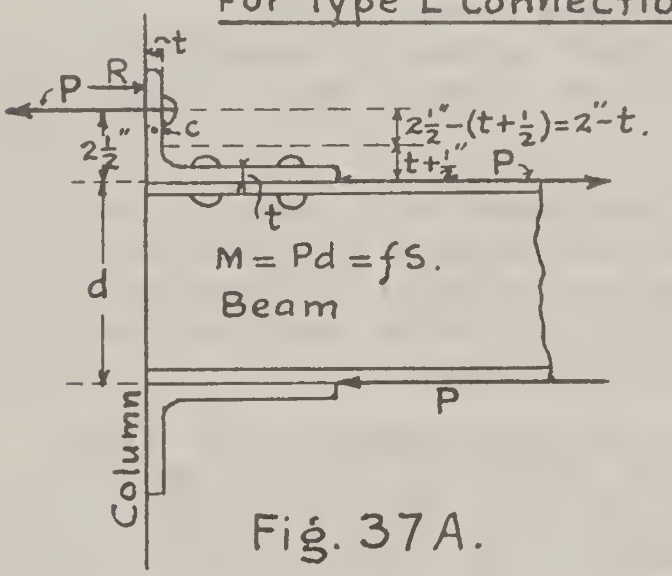

Fig. 37A.

Let

ℓ = length of connecting angle.

t = thickness of angle metal

P = tension or shear value of a groupe of 4 rivets

Assuming a point of contraflexure at c, then the bending moment in the angle leg will be $\frac{P}{2}(2-t)$, and this must be equal to the resisting moment $\ell t^2/6$ multiplied by the unit stress $f = 24000$. Hence the following equation

$$\frac{P}{2}(2-t) = \frac{\ell t^2}{6} \times 24000, \quad or \quad P - \frac{Pt}{2} = 4000\,\ell t^2$$

which solved for t gives: $t = \sqrt{\frac{P}{4000\ell} + \left(\frac{P}{16000\ell}\right)^2} - \frac{P}{16000\ell}$ - - - - 37A.

For $\ell = 12''$, $P = 31{,}800$ lbs., $t = 0.66$ inch for 4 - 3/4 rivets.

$P = 43{,}200$ " $t = 0.70$ " for 4 - 7/8 rivets

$P = 56{,}400$ " $t = 0.75$ " for 4 - 1 rivets.

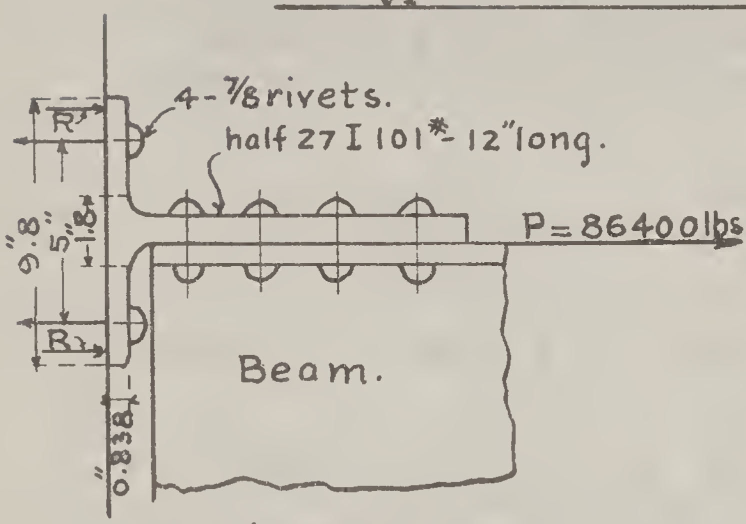

Fig. 37 B.

For type T connections, the heavy fillet is deducted from the span between rivets to obtain an effective span $S = 5 - 1.8 = 3.2$ inches.
The center moment of this span with restrained ends and due to a load P will be

$$M_{max} = \frac{Ps}{8} = \frac{86400 \times 3.2}{8} = 34200 \text{ in.lbs.}$$

The section modulus of the flange metal is $\frac{12 \times \overline{0.838}^2}{6} = 1.404$

giving a unit stress in bending of

$$f = \frac{M}{S} = \frac{34200}{1.404} = 24300 \text{ lbs. per sq. in.}$$

The case with a half 30" I 115 lbs, using 1" rivets or P = 112,800 lbs. gives a unit stress $f = 23900$ lbs. per sq. in.

Art. 38. Wind Pressure. The municipal building codes stipulate the wind pressure to be used in designing tall building frames, chimneys, sign boards, etc., but in the absence of any such requirements, the designer may have to decide the question on the basis of Weather Bureau observations.

In deducing wind pressures from velocity observations many difficulties are encountered even if the velocity were known with accuracy. Wind normally consists of a series of gusts, the speed and direction of which vary within wide limits.

The U. S. Weather Bureau furnishes wind velocities for many stations in the country and covering a long period of time. Since maximum velocities occur only at intervals of many years it is essential to select the extreme maximum for a ten to twenty year record for any locality.

The Weather Bureau velocities are somewhat too high according to experiments conducted at the U. S. Bureau of Standards, published in Scientific Paper No. 523 by Dryden and Hill. The true velocity was found to be 76% of the Weather Bureau velocity.

Records show velocities ranging from 50 to 120 miles per hour for various stations in the U. S., but it would not seem prudent to figure less than 60 miles even though weather reports showed a lesser maximum.

Except in some of the storm swept areas on the Atlantic Coast and the Gulf of Mexico, 100 mile wind is quite conservative and 87 mile wind is a fair maximum for all inland points.

The generally accepted formula for converting velocity into pressure is that of E. Eiffel, as translated by Hunsaker, U. S. Navy, 1913,

$$w = 0.0033\, V^2 \text{ lbs. per sq. ft} \qquad \text{38A.}$$

where V is wind velocity in miles per hour.

Whether the wind pressure requirements should be varied to suit the ratio of height to width of a building is questionable, because there are more important factors to be considered besides this ratio.

For strength alone, the specified unit stresses govern every design. However, rigidity is of even greater importance than strength, and this property depends on the dimensions of the frame and the method of carrying the wind loads.

On the assumption that the bare frame should sustain the entire wind load without exceeding certain allowable unit stresses, then the most rigid frame will be one of rather wide base compared with its height, and the wind shears are carried by diagonal members in direct stress. This, however, cannot be realized except in very rare cases, because diagonal members must be concealed in hollow walls which latter are very objectionable from the architectural standpoint.

In the usual type of tier building construction, in which the frame consists of beams and columns without diagonals, wherein all wind shears must be resisted by bending, the inherent rigidity of the frame is due to the number of columns of given story heights, and not to the ratio of height to width of frame alone. Of course a wide building usually contains more columns than does a narrow building, nevertheless, it is the number of columns and the rigidity of the beam and column connections which determine the rigidity of the frame as a whole.

The very considerable contribution to rigidity attributable to the architectural fill materials will usually afford ample security against excessive deflections as mentioned in Art. 31.

The practice of introducing an indeterminate factor, by assuming the upper six or eight stories of a tall building as contributing no wind load because of the architectural fill material, is not considered wise.

Chapter 8. Approximate Method of Designing Continuous Concrete Beams.

by
David A. Molitor, C. E.

Art. 39. Description of Method. The method here advocated for the design of continuous concrete beams is applicable only in building constructions where beams are of T shape and are cast monolithically with the columns. Such beams involve a large element of uncertainty which cannot be appraised by any practicable theoretical analysis.

The method consists of designing continuous beams like cantilever systems, in which the more or less arbitrarily located hinged points are treated as points of contraflexure. To make some allowance for shifting positions of the live load, the design moments are increased 20% over the actual theoretical moments of the cantilever system. The method has been used in Detroit, Michigan, for over ten years in the design of reinforced factory and office buildings and has proven eminently satisfactory.

The formulas for cantilever beams will first be given, together with a table of moment coefficients for end and interior spans. An illustrative problem will then be presented followed by a discussion of the method.

For any case of uniform loading the sum of the end and center moments for an interior span will equal $w\ell^2/8$, and adding 20% to each will give the sum of the negative end and positive center moments of $1.2 \times 0.125\, w\ell^2 = 0.15\, w\ell^2$. For end spans this sum is not a constant but increases as the point of contraflexure moves farther out from the restrained end.

The formulas are given in Table 39-A, and may be written out directly from Fig. 39-A. The formulas in the left half of the table represent the theoretical values, and those in the right half are working values increased by 20% to allow for variations in the points of contraflexure as affected by different cases of loading.

It should be noted that the locus of the point of contraflexure for an interior span is a parabola representing the constant sum moment $0.15\, w\ell^2$, while for an end span the locus of the point C is a straight line, such that for every point C, there will be a separate parabola passing through it to represent the end and center moments.

Fig. 39-A is drawn for one particular set of moments, with the moment over a support representing the end moments of any two adjacent spans. The same feature may be preserved even when the spans are of unequal lengths or when the loads on the spans are widely different. With this arrangement, no moment is transmitted to the column between any two spans.

CONCRETE CANTILEVER BEAMS, UNIFORMLY LOADED, DETROIT CODE.

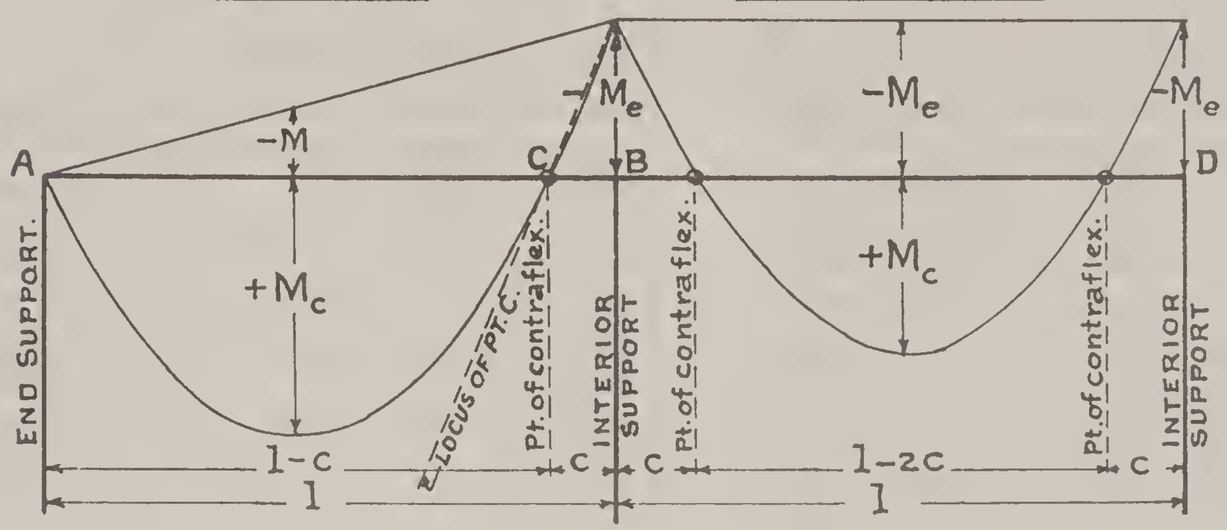

Fig. 39 A

FORMULAS.

MOMENTS FOR ALL SPANS LOADED

INTERIOR SPANS.

$$M_c = \frac{w}{8}(l-2c)^2$$

$$M_e = \frac{wc}{2}(l-2c) + \frac{wc^2}{2} = \frac{wc}{2}(l-c).$$

$$M_c + M_e = \frac{w\,l^2}{8} = \text{constant}.$$

$$c = \frac{l}{2} - \sqrt{\frac{l^2}{4} - \frac{2M_e}{w}} = \frac{l}{2} - \frac{1}{2}\sqrt{\frac{8M_c}{w}}$$

END SPANS.

$$M_c = \frac{w}{8}(l-c)^2 = \left(1-\frac{c}{l}\right)^2 \frac{w\,l^2}{8}$$

$$M_e = \frac{wc}{2}(l-c) + \frac{wc^2}{2} = \frac{wcl}{2}$$

$$M = \frac{M_e}{2l}(l-c) = \frac{wc}{4}(l-c)$$

$$M + M_c = \frac{w}{8}(l^2 - c^2)$$

$$M_c + M_e = \frac{w}{8}(l+c)^2 = \text{variable}$$

$$c = \frac{M_e}{0.5wl}$$

MOMENTS INCREASED 20%.

INTERIOR SPANS.

$$M_c = 0.15\,w(l-2c)^2$$

$$M_e = 0.6\,wc(l-c)$$

$$M_c + M_e = 0.15\,w\,l^2 = \text{constant}.$$

$$c = \frac{l}{2} - \sqrt{\frac{l^2}{4} - \frac{M_e}{0.6w}} = \frac{l}{2} - \frac{1}{2}\sqrt{\frac{M_c}{0.15w}}$$

END SPANS.

$$M_c = 0.15\left(1-\frac{c}{l}\right)^2 w\,l^2$$

$$M_e = 0.6\,wcl$$

$$M = \frac{0.6M_e}{l}(l-c) = 0.3\,wc(l-c)$$

$$M + M_c = 0.15w(l^2 - c^2)$$

$$M_c + M_e = 0.15\,w(l+c)^2 = \text{variable}.$$

$$c = \frac{M_e}{0.6wl}.$$

See Trans. Am. Soc. C.E, Vol. 91, p. 174, Paper entitled "Distribution of Reinforcing Steel in Concrete Beams and Slabs", by Boyd S. Myers, and Discussion on p. 212, by D. Molitor.

D. Molitor, C.E

Oct. 2, 1926.

TABLE 39A.

END AND CENTER MOMENTS FOR CONTINUOUS BEAMS.

END SPANS				
END MOMENT		CENTER MOMENT		POINT
Fraction	Decimal	Fraction	Decimal	Contra-flexure
WL^2	WL^2	WL^2	WL^2	L
1/19.2	0.0522	1/8	0.1250	0.087
1/18.5	0.0540	1/8.05	0.1242	0.090
1/17.5	0.0570	1/8.15	0.1229	0.095
1/16.7	0.0600	1/8.25	0.1215	0.100
1/15.9	0.0630	1/8.33	0.1202	0.105
1/15.1	0.0660	1/8.43	0.1188	0.110
1/14.5	0.0690	1/8.52	0.1175	0.115
1/13.9	0.0720	1/8.62	0.1162	0.120
1/13.3	0.0750	1/8.73	0.1148	0.125
1/12.8	0.0780	1/8.82	0.1135	0.130
1/11.9	0.0840	1/9.04	0.1109	0.140
1/11.1	0.0900	1/9.24	0.1084	0.150
1/10.4	0.0960	1/9.38	0.1058	0.160
1/9.80	0.1020	1/9.69	0.1033	0.170
1/9.25	0.1080	1/9.92	0.1009	0.180
1/8.78	0.1140	1/10.15	0.0985	0.190
1/8.33	0.1200	1/10.4	0.0960	0.200
1/6.67	0.1500	1/11.8	0.0845	0.25
1/5.55	0.180	1/13.6	0.0735	0.30
1/4.76	0.210	1/15.7	0.0635	0.35
1/4.16	0.240	1/18.5	0.0540	0.40
1/3.70	0.270	1/22.0	0.0454	0.45
1/3.33	0.300	1/25.7	0.0375	0.50
1/2.78	0.360	1/41.6	0.0240	0.60
1/2.38	0.420	1/74	0.0135	0.70
1/2.08	0.480	1/167	0.0060	0.80
1/1.85	0.540	1/666	0.0015	0.90
1/1.67	0.600	$1/\infty$	0.0000	1.00

INTERIOR SPANS.				
END MOMENT		CENTER MOMENT		POINT
Fraction	Decimal	Fraction	Decimal	Contra-flexure
WL^2	WL^2	WL^2	WL^2	L
1/40	0.0250	1/8	0.1250	0.045
1/35.1	0.0285	1/8.25	0.1215	0.050
1/32.0	0.0312	1/8.40	0.1188	0.055
1/29.4	0.0340	1/8.63	0.1160	0.060
1/27.4	0.0365	1/8.82	0.1135	0.065
1/25.6	0.0391	1/9.03	0.1109	0.070
1/24.1	0.0416	1/9.23	0.1084	0.075
1/22.7	0.0440	1/9.45	0.1060	0.080
1/21.5	0.0467	1/9.68	0.1033	0.085
1/20.3	0.0492	1/9.93	0.1008	0.090
1/19.4	0.0516	1/10.2	0.0984	0.095
1/18.5	0.0540	1/10.4	0.0960	0.100
1/17.7	0.0564	1/10.7	0.0936	0.105
1/17.0	0.0587	1/10.9	0.0913	0.110
1/16.4	0.0610	1/11.2	0.0890	0.115
1/15.7	0.0634	1/11.5	0.0866	0.120
1/15.3	0.0655	1/11.8	0.0845	0.125
1/14.7	0.0678	1/12.1	0.0822	0.130
1/14.3	0.0700	1/12.5	0.0800	0.135
1/13.8	0.0723	1/12.9	0.0777	0.140
1/13.4	0.0744	1/13.2	0.0756	0.145
1/13.1	0.0765	1/13.6	0.0735	0.150
1/12.7	0.0786	1/14.0	0.0714	0.155
1/12.4	0.0805	1/14.4	0.0695	0.160
1/12.1	0.0828	1/14.9	0.0672	0.165
1/11.8	0.0846	1/15.3	0.0654	0.170
1/11.6	0.0866	1/15.8	0.0634	0.175
1/11.4	0.0875	1/16.0	0.0625	0.177

D. Molitor.

The center moment need never exceed $WL^2/8$ for any span.
For Interior Spans the center mom. shall be not less than $WL^2/16$, and the end mom. shall be not less than $WL^2/40$.
For End Spans the end moment shall be not less than $WL^2/20$, and may be anything depending on adjacent spans.

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Oct. 2, 1926

Table 39-A is self explanatory and will suffice for all spans of variable lengths and loadings.

In the following example no attempt is made to delve into the theory of reinforced concrete design, but merely to illustrate the method of arriving at a suitable assignment of end and center moments in a series of continuous beams. The standard nomenclature used will be found in any hand book on reinforced concrete.

Art. 40. Example 40-A. Three spans have uniform loads of 2200 lbs. per lin. ft. and concentrated loads as shown in Fig. 40-A, to design the beams on the basis of the following unit stresses: $f_c = 750$ lbs per sq. in., $f_s = 20,000$ lbs. per sq. in., $n = 15$, $R_c = \frac{1}{2} f_c kj = 118.8$, and the moment of resistance = M.R. $= \frac{118.8}{12} bd^2$ ft. lbs. for concrete sections. The allowable unit shear is not to exceed 120 lbs. for a section with bent bars and stirrups. The beam has a 4 inch solid slab flush with the top and will act as a T beam. The end shears for each span are given on the diagram.

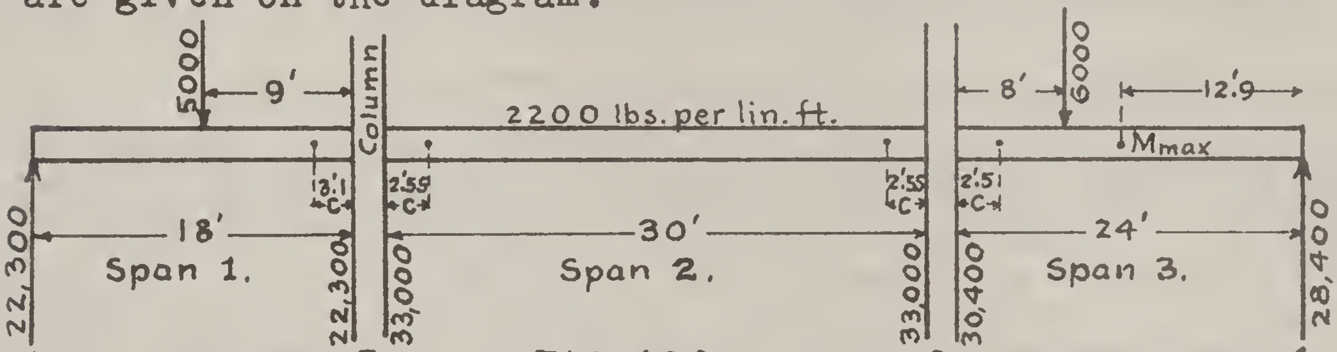


Fig. 40A.

First choose a beam section adequate for the maximum end shear of 33,000 lbs. which occurs in the center span. For this shear make the beam 12" x 30" giving a unit shear $v = \frac{33000}{0.88 \times 12 \times 28} = 112$ lbs. Then M.R. $= \frac{118.8}{12} bd^2 = -93,100$ ft. lbs. for the stem without using any compression steel. This moment requires a tension steel area $A_s = \frac{M \text{ ft. lbs.}}{1460\, d} = \frac{93100}{1460 \times 28} = 2.27$ sq. in.

for which use 3 - 1 in. round bars.

This section will be accepted for all three spans, making all end moments equal to 93,100 ft. lbs. and the corresponding center moment for each span will now be found using the moment coefficients given in Table 39-A. A T beam section of 33 in. width of T will be ample for the max. $M_c = 204,000$ ft. lbs.

End Span 1. Max. $M = 2200 \times \frac{18^2}{8} + 5000 \times \frac{18}{4} = 111,800$ ft. lbs.

For this moment make $w\ell^2 = 8 \times 111,800 = 894,000$ ft. lbs.

For an end moment $M_e = \frac{w\ell^2}{x} = 93,100$, find $x = 9.65$

From the table for end spans, for an end moment with $x = 9.65$ find the corresponding center moment with $x = 9.75$ and $c/\ell = 0.172$

Hence $M_c = \frac{894,000}{9.75} = 92,500$ ft. lbs., and $c = 0.172\,\ell = 3.1$ ft.

The required steel area $A_s = \frac{92500}{1460 \times 28} = 2.26$ sq. in = 3 — 1" round bars. Make one straight (full length); one bent for shear; and one bent for double continuity over column 2,

expecting to get two more 1" round continuity bars from span 2.

Int. Span 2. Max. $M = \frac{w\ell^2}{8} = 247{,}500$ ft. lbs. and $w\ell^2 = 1{,}980{,}000$ ft. lbs.
For an end moment $M_e = \frac{w\ell^2}{x} = 93{,}100$ ft. lbs., find $x = 21.3$
and the center moment $M_c = \frac{1{,}980{,}000}{x = 9.3} = 204{,}000$ ft. lbs. and $c = 0.085 \times 30' = 2.55$ ft.
The required steel area $A_s = \frac{204\,000}{1460 \times 28} = 5.0$ sq. in. Use 2 - 1 in. square bars (full length); two 1 in. round bars bent for shear; and 2 - 1 in. round bars bent for double continuity at each end of span 2.

Figuring a 4 inch T, 33 in. wide, the moment of resistance of the T beam = 204000 ft. lbs. as required.

End. Span 3. Max. $M = 28{,}400 \times 12'.9 - 2200 \times \frac{12.9^2}{2} = 184\,000$ ft. lbs.
For this moment make $w\ell^2 = 8 \times 184{,}000 = 1{,}472{,}000$ ft. lbs.
For an end moment $M_e = \frac{w\ell^2}{x} = 93100$, find $x = 15.95$.
From the table for end spans, for an end moment with $x = 15.95$ find the corresponding center moment with $x = 8.33$.

Hence $M_c = \frac{1{,}472{,}000}{8.33} = 177{,}000$ ft. lbs.; and $c = 0.105 \times 24' = 2.5$ ft.

The required steel area $A_s = 4.33$ sq. in. Use 2 - 1 in. square bars and 3 - 1 in. round bars. Make 2 - 1" square bars straight; 2 - 1" round bars bent for shear; and 1 - 1" round bar bent for double continuity over column 3.

The bars bent for double continuity are bent up at the point of contraflexure at 45° and extend into the adjacent spans 1 ft. beyond the respective points of contraflexure nearest the column support. It will be seen that the foregoing arrangement of bent bars furnishes three 1 in. round bars over the top of each support as required for the end moments of 93,100 ft. lbs. The bars bent for shear are bent up at 45° next to the columns and at the outer ends and are hooked into the columns and at the outer ends.

No extra bars are thus required for the negative end moments as the necessary steel was obtained by proper bending and careful selection of bar sizes to make the steel areas correspond with the computed areas at all beam sections. Also, the end moments on each side of each column being equal, there will be no moment transmitted to the columns except a very nominal moment which may result from partial live loads on any of the spans.

In the foregoing example, say 2.0 sq. in. of compression steel might have been utilized by extending the bottom straight bars into the columns and adjacent spans, making the negative end moments 129,000 ft. lbs. instead of only 93,100 ft. lbs. as furnished by the stem only. This would have resulted in somewhat smaller center moments and a slight economy in steel.

If, for any reason, the end moments of an interior span are not made equal, a condition which is governed by the lengths and loadings of the adjacent spans, then the center moment is based on the mean value of the two unequal end moments. Similarly, if a certain end moment is assigned at Col. 1 or Col. 4, to be transmitted from the end spans to the end columns, then such end spans are designed for center moments based on the average end moments of these spans, the same as for interior spans with end moments.

The shear reinforcement still to be provided for in the form of stirrups is computed on the assumption that the concrete is permitted to take 40 lbs. of shear per sq. in. of stem area, and the bent shear bars are good for the added shear increment between the point of the first bend and the end of the span.

Thus for span 2, the gross end shear is 33000 lbs. the stem area takes 40 bjd = 11,760 lbs. and the bent bars will carry 3 x 2200 = 6,600 lbs. leaving a net shear V=14,640 lbs. for which stirrups must be provided.

In presenting this example, sufficient descriptive matter was deemed necessary to illustrate the use of the table. In practice, however, the complete numerical work can be reduced to about six lines for the 3 span problem.

Art. 41. Justification of the Method. On strictly theoretical lines there is some objection to this method of designing reinforced concrete beams with continuity, on the presumption that beams so designed must function strictly in accord with the theory of elasticity. Yet no tests made to date prove that reinforced concrete beams, designed otherwise than by the theorem of three moments, have proven unsatisfactory in service, and such service covers periods of ten to fifteen years on many large and important buildings.

No objection, other than theoretical, has ever been raised against this method of design, and perhaps too much weight has been given to matters which are beyond the possibility of correct theoretical analysis. If such beams have proven satisfactory under normal working conditions in buildings, that fact should be accepted as a test result, irrespective of what would happen under tests carried to ultimate failure.

In practically all tests made on continuous concrete beams designed according to the elastic theory, the steel over the supports slipped to some extent and was never fully stressed even when forced into action by the positive moment, indicating that some extra steel should be supplied at the center of the span where slipping cannot occur to make up for the loss at the ends.

In reinforced concrete buildings, the beams are T shaped and monolithic with floor slabs and columns, with reinforcing steel so interlocked as to prevent unsightly

cracks. The superimposed loads are of extremely variable character, uniform dead and live loads and various concentrated loads, all of which may attain the maximum values assumed in the design, or they may be greatly reduced when there is no live load. It would be futile to attempt a rigid analysis of such a beam, or system of beams, in accordance with the theory of elasticity and hence the need for a simple approximate method.

A T beam subjected to negative end moments is good for no more than the resisting moment of the stem at the ends, while the center portion carrying the positive moment, may have double this capacity in compressive area of the slab which forms the T. Therefore, it is wasteful to bolster up the end section by haunching or using compression steel, which must be done if the end section is to carry more moment than the center section.

By ignoring the theoretical moment distribution indicated for continuity, and treating the chain of spans as a series of cantilever systems, in which the hinged points represent arbitrarily assigned points of contraflexure, the above mentioned difficulties are obviated.

Now it may be contended that, because the points of contraflexure are treated as imaginary hinges of a cantilever system, which hinges have not actually been provided in the construction, that it is not certain that the structure will behave in accordance with these assumptions.

Anyone of the various assumptions that may be made as to moment distribution in this analysis, represents possible conditions for some cantilever system, and if the material (both steel and concrete) at any section of a beam is ample to care for the moment assigned to it, each such section is capable of contributing its share of total internal work to be performed.

The possible shift of a point of contraflexure can be reduced to a minimum limit by bending up part of the positive steel so as to have the first bent bar intersect the beam axis in the point of contraflexure and extend out into the adjacent span for a distance $d/2$ beyond the adjacent point of contraflexure. This is called bending for double continuity (D. C.) Other bars bent merely for shear, may also act for continuity if allowed to extend beyond the supports into adjacent spans. This is called bending for shear and continuity (S. & C.)

Since the loads applied to a beam will first stress the central portion containing the positive steel, before the load effect can reach the supports, it follows that the positive steel must work to capacity, delivering to the cantilever ends only that portion of the load in excess of its capacity and for which the negative steel makes ample provision.

On the other hand, enough negative steel may be provided to carry the entire load on cantilever arms, thus unloading the central portion where little or no positive steel would then be required. However, this alternative is not utilized except in special cases of short interior spans or short end spans, when the end moments from the longer adjacent spans, are proportionately large.

While continuous beams, designed according to the theory of elasticity, develop moments approximating the designed values within the range of proportionality, that does not prove (as many assume it does) that for beams designed otherwise the results under tests would disagree materially with the assumptions on which the design was based. In fact, the few tests reported by Prof. Moersch distinctly support this latter view. Although the behavior of the beam within the range of proportionality now becomes somewhat different, the ultimate behavior at rupture again agrees with the variant assumptions.

After the elastic limit is exceeded, the elastic theory no longer governs, and entirely new relations are set up. For this there is abundant evidence which has not received proper consideration by theorists.

Ordinary steel elongates about 25% of its length at the ultimate stress, while the elongation at the elastic limit amounts to less than 0.1%. If then some of the steel which is stressed beyond the elastic limit is permanently stretched 1% due to overloads which have stressed steel in other parts of the structure involving redundancy merely to the elastic limit, then a readjustment of stress will take place resulting in all steel stressed equally. In the case of a reinforced concrete beam all this can take place without showing visible cracks over the supports. When cracks do appear it is due to much greater overloads, approaching ultimate stress, or to other causes attributable to methods of construction.

Advantage can be taken of the ductility of steel only when dealing with structures involving redundancy, and since continuous beams belong to this class, it is proper to give consideration to this steel characteristic.

Take, for instance, a redundant tension member in a steel frame or truss, and suppose that its length is shorter than intended, and the member is forced into place and its ends are then permanently connected. When the frame receives its maximum load, this particular member will be overstressed before the other members attain their calculated stresses. The stress in such a redundant member may readily exceed the elastic limit, and if it does, this member simply stretches a small amount (permanently) and thus adjusts itself to the defects in construction without producing any permanent damage. Thereafter, for repeated applications of the maximum load, the member will function normally without ever again being stressed beyond its yield point.

This is a well known phenomenon, but, strangely enough, it has received little consideration in practice. A better understanding of this subject would relieve most doubts now entertained by many engineers with regard to statically indeterminate structures.

The following statement by F.v. Emperger in Beton u. Eisen, June 20, 1930, Vol.29, No.12, p.216, confirms the above conclusions. "The moment curve can be exactly determined only for simply supported beams. Where restraint exists the moment variation may be studied by considering possible limiting conditions of restraint. The usual assumption of point supports, satisfactory in bridge work, does not seem reliable for buildings where the columns act with the beams".

"Building Departments of Leipzig and Hamburg have now (1929) approved procedures along new lines. Safety against failure should be the governing factor in beam design. The sum of the center and support moments rather than moment at any point is to be considered. Attainment of the elastic limit at any point leads to a readjustment of the moment curve, causing higher moments at other portions of the beam. In concrete, flexibility of design possible in reinforcement and haunching, permits original moment distribution between center and supports to exist until failure occurs."

"In tests by the Austrian Reinforced Concrete Committee, (Heft 4, 1913) a fully restrained concrete beam was reinforced for $\frac{wl^2}{24}$ at supports and $\frac{wl^2}{12}$ at the center. At failure, the moments had adjusted themselves to fit this steel distribution."

"At some future date it is hoped to continue tests of duplicates of single span, restrained beams already studied, when such beams are parts of rigid frames."

The procedure advocated in this chapter, for the design of reinforced concrete beams, may be restated as follows:

For continuous beams treated as cantilever systems, the end and center moments may be proportioned as follows:

For intermediate spans, the end moment shall never be less than $\frac{wl^2}{40}$, and the center moment shall never be less than $\frac{wl^2}{16}$, while the sum of the mean end and center moments shall be not less than $0.15wl^2$.

For end spans, the end moment shall be not less than $\frac{wl^2}{20}$ and may have any value greater than this, depending on the end moment of the adjacent span. The sum of the end and center moments for an end span shall be not less than $0.15\ w(l + c)^2$, where $c = \frac{\text{end moment}}{0.6\ wl}$ = the distance from the fixed end to the point of contraflexure of the end span.

CHAPTER 9 - RETAINING WALLS.

Art. 42 Earth pressure on the back of a wall. It is not proposed to enter upon a theoretical discussion of this subject but merely to present, in brief form, the most usable features of the many theories evolved in the past by Coulomb, Poncelet, Scheffler, Mohr, Rankine, Winkler, von Ott, Weyranch, Mueller-Breslau and others. Of these may be mentioned Coulomb's theory of the prism of maximum pressure and the theory of conjugate stresses by Rankine, as the two outstanding ones, the combined features of which will be employed in the following outline, together with Poncelet's graphic solution.

When an earth embankment is artificially produced by dumping the material loosely from dump carts or flat cars, the material assumes a natural surface which is usually not a plane, nor does the averaged plane of slope make a constant angle ϱ with the horizontal as is commonly assumed. Actually the angle varies, becoming smaller with increased height of embankment. For certain clay materials the angle varies from 45° for heights up to 6 ft. and reduces to about 18° for a height of 65 ft. See paper by the author* "The Present Status of Engineering Knowledge Respecting Masonry Construction." For clean sand the conditions are most favorable and the angle of repose ϱ varies only slightly for a wide range of height. Other materials manifest a behavior intermediate between sand and clay. Laboratory experiments on dry material have no value owing to miniature proportions and doubtful degree of moisture which a given material should possess when exposed to the elements for a long period of time.

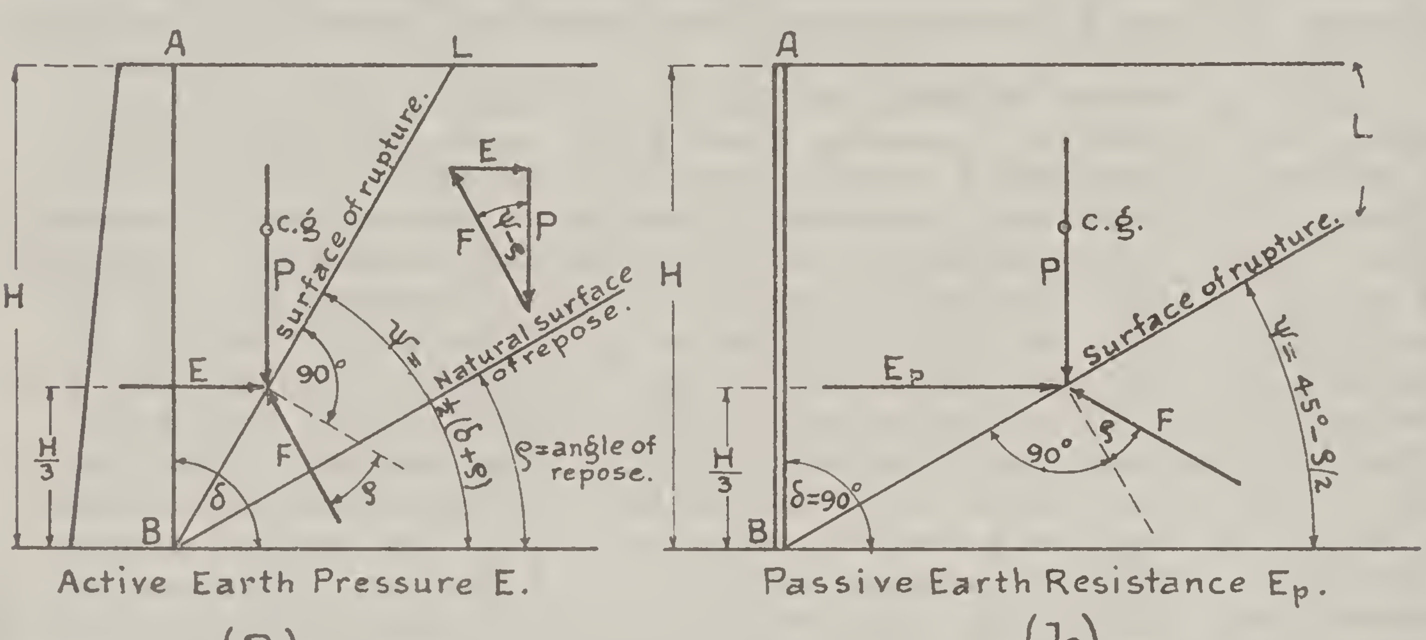
When an embankment is filled to a steeper slope than is naturally possible for the material to maintain permanently by internal friction, then a slide will occur and the surface slope will adjust itself to a flatter angle. While this readjustment is taking place the surface material will move on a surface of rupture formed in the interior of the fill. The material beneath this surface of rupture will remain undisturbed, while the original surface slope will gradually assume a position of rest approximating the angle of repose. The surface of rupture is never a plane, being approximately hyperbolic for clay and approaching a plane for clean dry sand. However, a plane surface of rupture is usually assumed in dealing with the subject along theoretical lines.

With these introductory remarks, based on many years of careful observation, it should be apparent that any attempt at a purely theoretical analysis of this problem is quite futile and the most that may be accomplished is to arrive

*Journal Assn. Eng. Soc. 1900, p. 31 containing a new Theory of Earth Pressure for Clay.

at a rational approximate method devoid of unnecessary mathematical manipulations. The natural surface of repose and the surface of rupture will be assumed as plane surfaces in the following presentation.

Art. 43 Earth Pressure on walls without surcharge. Fig. 43-A represents the case of active earth pressure (a), and that of passive earth resistance (b), considering these forces without regard to the stability requirements to be fulfilled by the wall structure itself.



Figs. 43A.

The active earth pressure E is the resultant pressure which the wall AB, Fig. a, must exert against the wedge ABL to prevent sliding on the plane of rupture BL, when the wedge is acted upon by its own weight P. The resultant pressure F, making the angle ρ with the normal to the surface of rupture, represents the force necessary to resist the wedge pressure from below. The three forces P, E and F must be in equilibrium as indicated by the triangular force polygon. The maximum value of E occurs when the angle $\psi = \frac{1}{2}(\delta + \rho)$, which signifies that the surface of rupture bisects the angle $\delta - \rho$, which is the angle between the surface of repose, and the surface AB or back of the wall. In this case the pressure wedge ABL must become active before the wall is called upon to offer the maximum resistance E.

According to Coulomb the horizontal componentof the active earth pressure E for $\delta = 90°$, is given by the formula

$$E = w H^2 \tan^2 (45° - \varrho/2) \text{ --------------- } \quad 43\text{-A}$$

where H is the height of wall and w is the weight per unit volume of the back fill.

<u>The Passive Earth Resistance E_p</u> is the maximum force which the earth prism ABL, Fig. b, can just resist without being displaced by sliding upward on a certain plane of rupture BL, thus lifting the load P and bringing into action the resisting force F on the plane of rupture.

The three forces E_p, P and F must be in equilibrium in the instant that any motion occurs.

It should be noted that the minimum value of E_p governs, since the earth fill in resisting this force passively must do so without moving. The passive resistance E_p attains its minimum value when the angle $\psi = \frac{1}{2}(\delta - \varrho)$.

According to Coulomb the horizontal component E_p, of the passive resistance, when $\delta = 90°$, is given by the formula

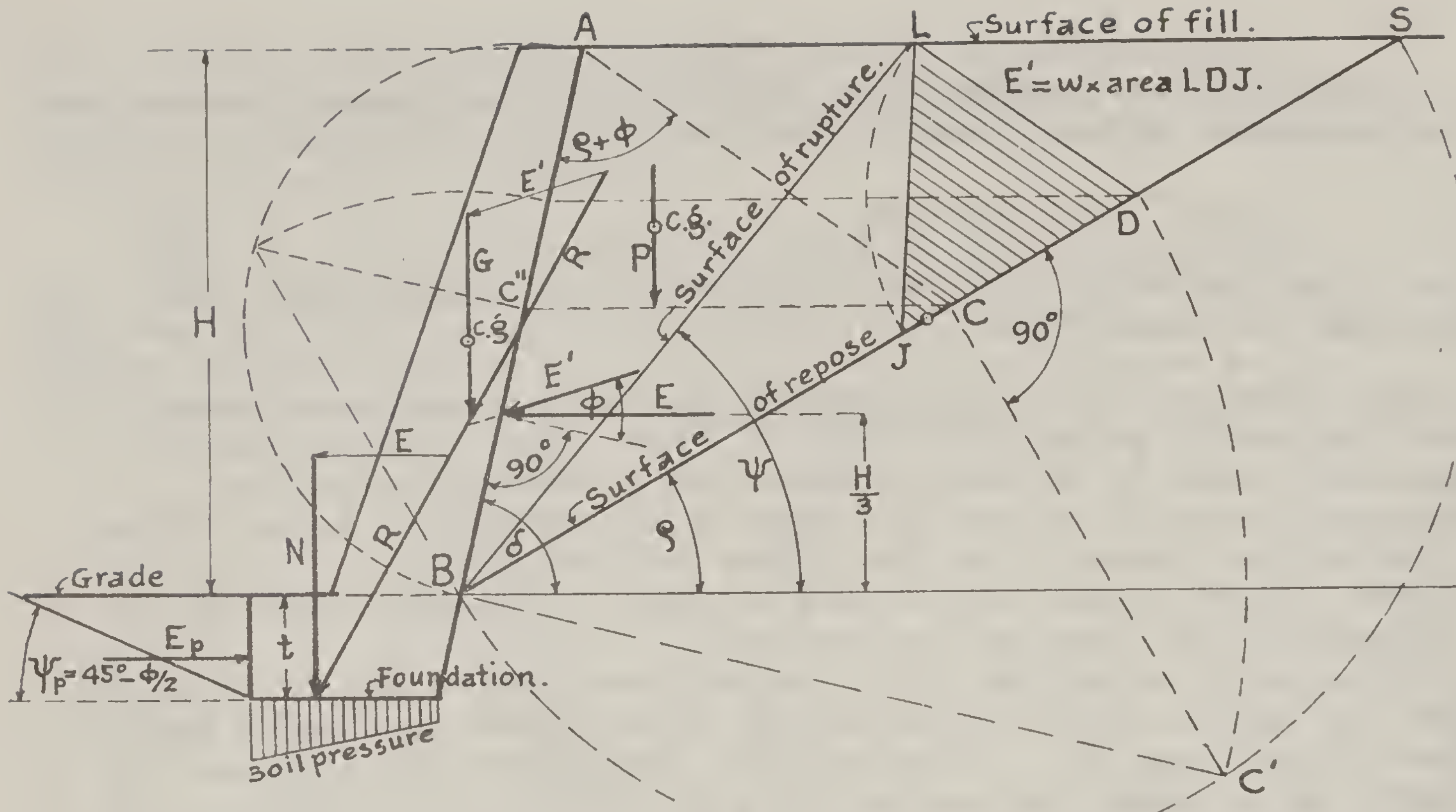
$$E_p = w H^2 \tan^2 (45° + \varrho/2) \text{ ----------------- } \quad 43\text{-B}$$

<u>Art. 44</u> <u>Graphical Solution, Wall without surcharge.</u> The actual force representing the active earth pressure on the back of a wall is not the horizontal force E previously found, but a force E′ of which E is the horizontal component. The force E′ makes a certain angle ϕ, with the normal to the back of the wall, and this angle is now the angle of friction between the earth fill and the wall surface, instead of the angle of internal friction of the fill material. The angle ϕ is always smaller than ϱ and depends on the nature and roughness of the wall as well as on the fill material.

Fig. 44-A will serve as a general picture of the several dimensional relations and forces involved in this and succeeding problems. The question of stability of the structural element or wall, not previously considered, will be taken up later.

The method here given was developed by Poncelet and is applicable to practically all cases of walls inclined at any angle δ less or greater than 90°, and all cases with surcharge.

The inclination of the back of the wall has a marked effect on the active earth pressure E as given by Eq. 43-A which is true only for $\delta = 90°$, or nearly so. When δ is less than 90° the earth pressure is less than for vertical walls, and when δ is more than 90° the earth pressure is greater than that for vertical walls.



Graphical Construction according to Poncelet for E′ and ψ.

Fig. 44 A.

w = weight, pounds per cu. ft. of back fill material.

ς = angle of repose or angle of internal friction.

ϕ = angle of friction between fill material and the back of the wall.

δ = angle between the back of the wall and the horizontal.

ψ = the unknown angle which the surface of rupture makes with the horizontal.

$\psi_p = 45° - \phi/2$ = angle of surface of rupture for passive resistance.

H = height of wall above grade in feet.

P = weight of earth prism ABL.

G = weight of the wall section down to its foundation.

The Poncelet construction proceeds as follows: At A lay off the known angle $\varsigma + \phi$ and draw AC to the intersection C with the natural slope BS. Then describe a semi-circle with BS as diameter, and draw the line CC′ perpendicular to BC. Now with B as a center and a radius BC′ describe an arc to locate the point D. Finally draw DL // AC which locates the point L on the surface of rupture BL, thus determining the angle ψ . The length DL is transferred onto BS, giving the triangle L D J, the area of which multiplied by w is the active earth pressure E′ . The direction of E′ makes the angle ϕ with the normal to the back of the wall and its point of application is at H/3 above B.

The same solution may be conducted by describing a semi-circle on the diameter AB and proceeding as indicated by the dotted construction lines.

According to the Poncelet solution, the surface of rupture divides the area ABDL into two equal areas making area ABL equal to the area LBD.

The stability of the wall is found by combining E′ with the weight G of the wall, into the resultant R, acting on the base of the wall. The vertical component N of R produces the soil pressure on the base.

The horizontal component E of E′ must be resisted by the passive resistance E_p in front of the toe, plus the resistance to sliding on the base which is $N \tan \phi$. Allowing a factor of safety of two, the condition equation for safety against sliding becomes

$$2E \leqq E_p + N \tan \phi \text{, where } E_p = wt^2 \tan^2(45° - \phi/2) \text{ ---- } 44\text{-A.}$$

Art. 45 Graphical Solution, walls with surcharge.

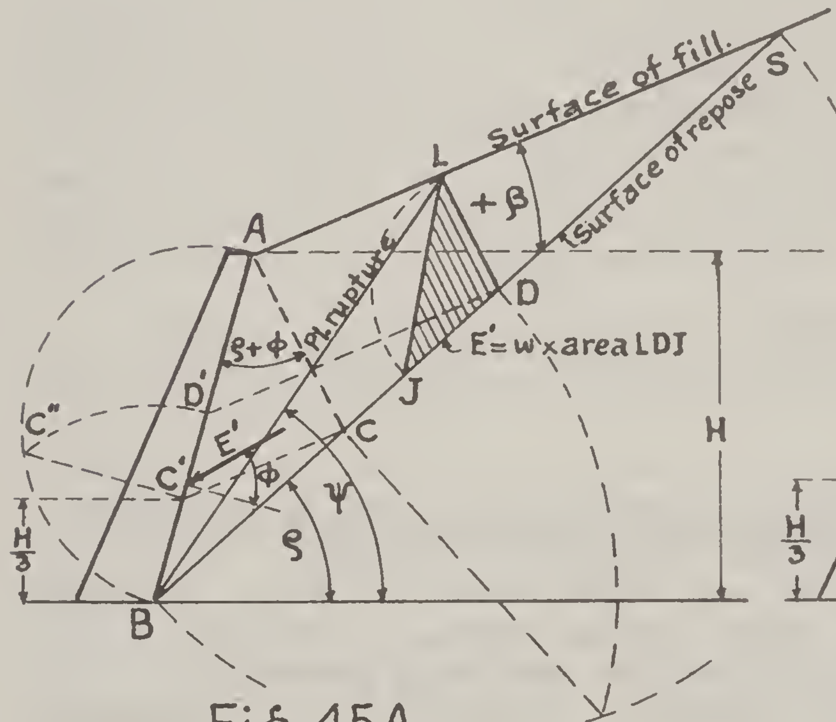


Fig. 45A.

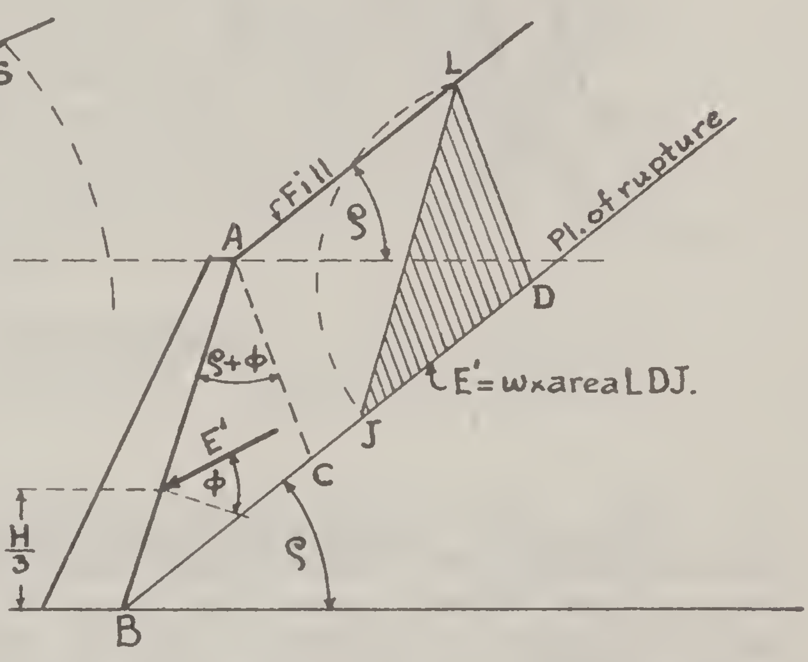


Fig. 45B.

When the surface of the fill is inclined at an angle β with the horizontal as in Fig. 45-A, the solution by Poncelet's method is precisely as given in Art. 44. By noting the lettered points, which are identical in both problems, there will be no difficulty in following the solution as given first for the circle on BS and then for the circle on AB. The active earth pressure is found as $E' = w \times \text{area } LDJ$.

The same construction is applicable when the angle β is negative and the line AS falls below the horizontal.

Fig. 45-B shows the case with surcharge inclined at the angle of repose making $\beta = \varrho$, where the point S is at infinity. The method now takes on a special form.

The natural surface of repose B D now becomes the surface of rupture and the solution consists simply in drawing the line AC with the known angle $\varrho + \phi$ at A and then constructing a triangle LDJ by drawing $LD \parallel AC$ and making $DJ = LD$. The active earth pressure is $E' = w \times \text{area } LDJ$.

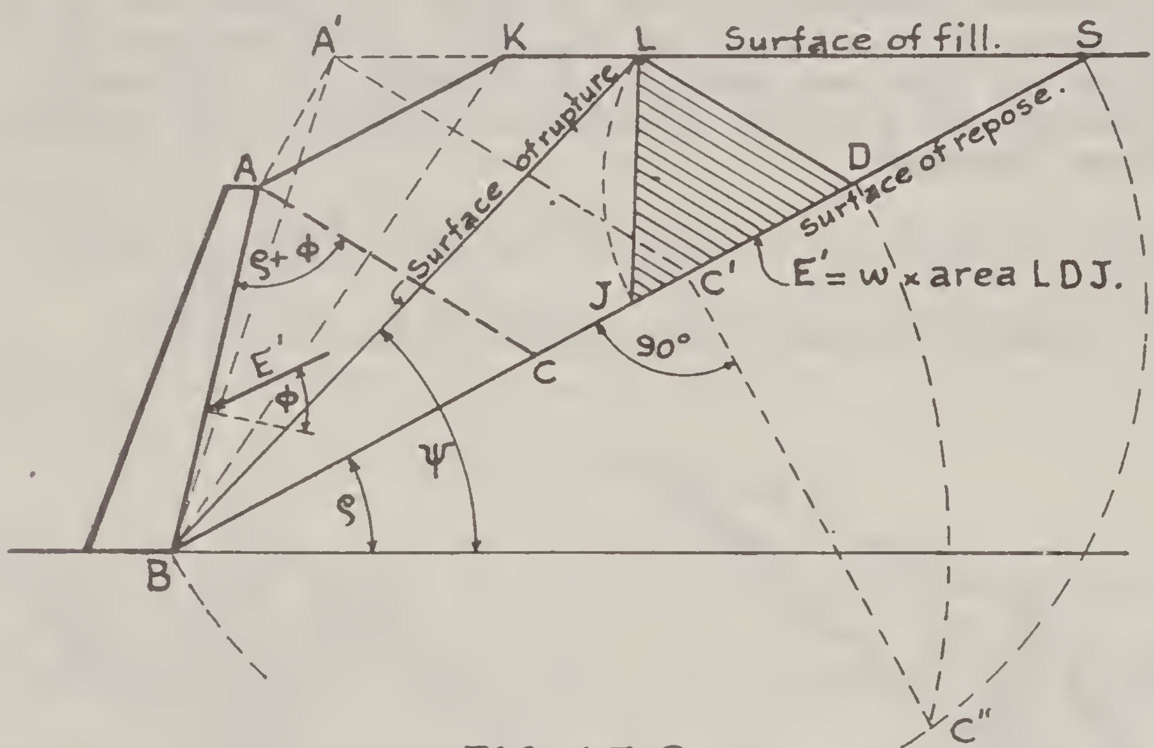


Fig. 45 C.

Fig. 45-C represents a broken surface line which requires some adjustment before applying the Poncelet Solution. To accomplish this, draw the line BK and then draw $AA' \parallel BK$ making the triangles ABK and A' BK of equal areas, with A' on the prolongation of KS. The construction is now completed as before except that point A' now governs instead of A, although A'C' still remains parallel to AC.

Art. 46 Practical Wall Design Data The various kinds of walls treated herein may be divided into gravity walls (Figs. a to d) and reinforced concrete walls (Figs. e to h) and the latter may be of the cantilever, counterfort or buttressed type. See Figs. 46-A.

In determining the earth pressure E on the back of any wall, the extent of the backfill material and its physical properties must be known, also whether or not the backfill can assist by its own weight in resisting a part of the overturning effect due to the earth pressure. A surcharge of any kind, superimposed on the back-fill, will increase the earth pressure and must be considered in the formula for E. The surcharge is represented by a height h of fill material.

The stability of a wall depends on two conditions: first, to resist the overturning moment of the active earth pressure, and also to resist the sliding on the base plane of the wall due to the horizontal component E of the active earth pressure.

It is, therefore, necessary to determine this horizontal component E and also the vertical component N of all forces on the wall above its base plane, in testing the stability against overturning and sliding of any wall of given type or design.

The earth pressure acting on the back of a wall, and tending to move the wall forward, is called the active earth pressure E' and this has a horizontal component E to be evaluated presently. The passive resistance E_p is the force necessary to move a certain bank of earth against its natural sliding tendency and may be developed in front of a wall footing or in front of anchorages embedded in the earth. Passive resistance becomes important in dealing with a variety of problems to be considered later.

The horizontal component E of the active earth pressure on vertical walls, according to Rankine - Coulomb, is given by the formulas for

$$\left.\begin{array}{ll} \text{walls with level fill} & E = \frac{w}{2} H^2 \tan^2(45° - \varrho/2) = \frac{w}{2} H^2 \Theta = K H^2 \\ \text{walls with surcharge} & E = \frac{w}{2} H(H+2h)\tan^2(45° - \varrho/2) = \frac{w}{2}\Theta(H+2h)H = K(H+2h)H \end{array}\right\} \quad 46A$$

where w is the weight per cu. ft. of backfill and surcharge, and ϱ is the angle of repose of the material or angle of internal friction.

See Figs. 46-A for the several varieties of cases that may be treated with these formulas and consult Table 46-A for values of w, ϱ, $K = \frac{w}{2}\Theta$, $\Theta = \tan^2(45° - \varrho/2)$, and $\Theta_p = \tan^2(45° + \varrho/2)$, for various materials.

The resultant diagonal earth pressure E', is a force making a certain angle of friction ϕ with a normal to the back of the wall and having for its horizontal component the value of E from Formula 46-A. Values of ϕ are given in Table 46-A.

The point of application of E is at $r = H/3$ above the base for liquids, and for materials listed in Table 46-A it is taken at $r = 3/8\,H$ when the backfill is level with the top of the wall. When there is surcharge, the point moves up to a position shown on the diagrams, with

$$r = \frac{Hh + 3/8\,H^2}{H + 2h} \qquad \text{46-B}$$

The Resultant R acting on the base. The vertical loads consist of the weight of wall G including the footing, and the weight P of the backfill when there is a heel. These are combined into the resultant Q representing the entire load on the base. The resultant R, found by combining Q and E graphically, represents the total effect of all forces acting on the supporting base. The horizontal component of R is E, and the vertical component is N, acting on the base with an eccentricity ε measured from the center of the base.

The distribution of the vertical load N on the base is found from Navier's law giving the following toe pressures, for dimensions in feet:

$$\left.\begin{aligned} f &= \frac{N}{D}\left(1 \pm \frac{6\varepsilon}{D}\right) \text{ lbs. per sq. ft., when } \varepsilon < D/6\text{, see Fig. (a).} \\ f &= \frac{2N}{3c} \text{ lbs. per sq. ft., when } \varepsilon > D/6 \text{ or when } c < D/3\text{, Fig. (e)} \end{aligned}\right\} \quad \text{46-C}$$

The unit toe pressure f should never exceed the allowable soil pressure as given below, and with this proviso, the factor of safety against overturning about the toe may be expressed as

$$\gamma = \frac{Qx}{Er} \qquad \text{46-D}$$

where x is the horizontal distance from the toe to the resultant load Q, Fig. (e).

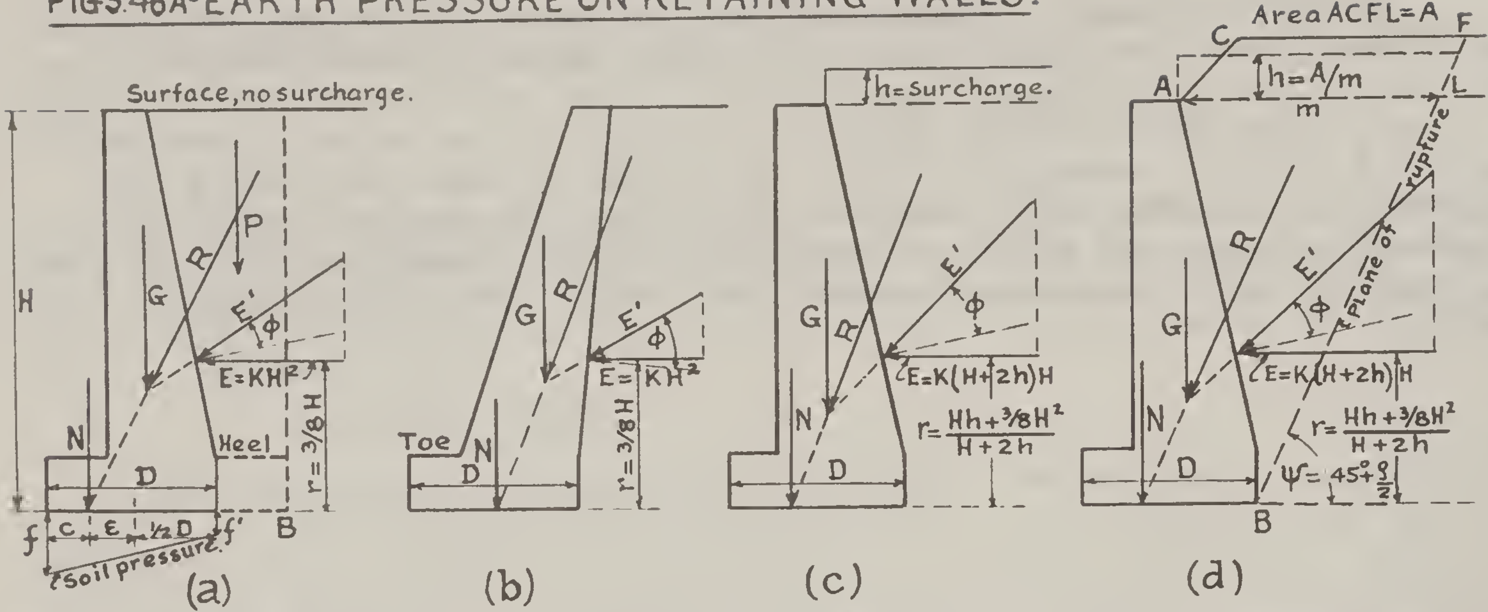
Safety against sliding on the base. The horizontal component E, of the earth pressure E', may be resisted on the base of the wall by friction on the foundation between the concrete and the earth; by passive earth pressure developed in front of the toe; and by a key under the footing as in Fig. (f). Any, or all, of these resisting forces may be called into action in certain cases, and the wall design should provide ample safety to resist a horizontal force of about 2 E.

The frictional resistance on the base plane may be taken as $N \tan \phi$, and the passive earth resistance at the toe as

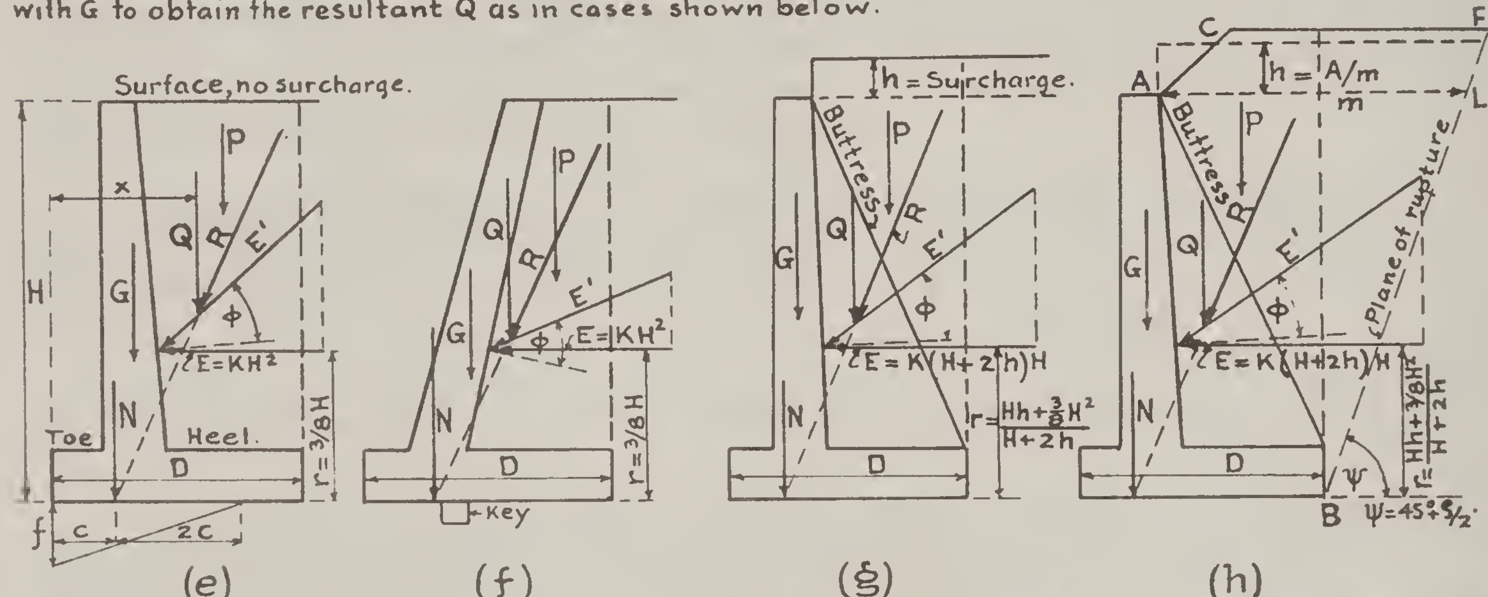
$$E_p \doteq \frac{w}{2} t^2 \tan^2 (45° + \phi/2) \qquad \text{46-E.}$$

where t is the height of fill in front of the toe and ϕ is the angle of friction between earth and concrete, Table 46-A. The resistance in front of the key may be taken at 0.7 of the allowable soil pressure under the base. See also Problem 47-C.

FIGS. 46A - EARTH PRESSURE ON RETAINING WALLS.



NOTE - When the base projects back as at B, Fig. a, then the weight of earth prism P is combined with G to obtain the resultant Q as in cases shown below.



FORMULAS - $E = \frac{w}{2} H^2 \tan^2(45° - \varphi/2) = KH^2$. UNIT TOE PRES. $f = \frac{N}{D}\left(1 \pm \frac{6e}{D}\right)$, FIG. a, OR $f = \frac{2N}{3c}$ WHEN $c < \frac{D}{3}$.

PASSIVE EARTH PRESSURE $= \frac{w}{2} h^2 \tan^2(45° + \phi/2)$. FOR TYPES FIG. e, f, g, h, APPROX. BASE WIDTH $D = 0.85(H+h)\sqrt{\theta}$.

FOR SAFETY AGAINST SLIDING ON THE BASE $N \tan\phi \geqq 2.0E$, USE A KEY, AS IN FIG. e, WHEN NECESSARY.

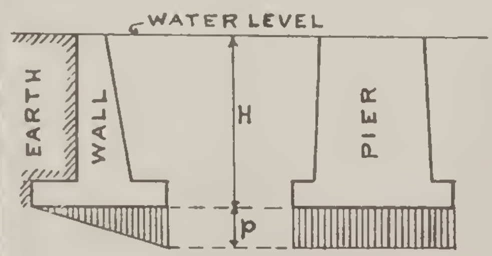
FOR VALUES OF ω, ς, ϕ, θ AND K SEE TABLE 46A. WALL SECTIONS ANALYSED FOR 1-FT. OF LENGTH.

ALLOWABLE SOIL PRESSURE AT TOE OF WALL

FOUNDATION MATERIALS	f lbs. sq. ft
Mud, quicksand, silt, soft clay.	piles
Clay, consistancy of putty, no visible water.	1000
Clay, stiff, moist, yielding slightly under thumb pressure.	2000
Clay, moist, dry enough to crumble.	5000
Sand in natural situ and compact.	4000
Sand and gravel, compact as in gravel pits.	6000
Sand clay and gravel, compact, no visible water.	6000
Coarse gravel, cemented with moist clay.	6000
Hard pan protected from water.	7000
Stratified clay protected from water.	8000
Slate and shale rock.	12000
Sandstone bed rock or better.	30000

HYDROSTATIC UPLIFT ON WALLS

FOUNDATION MATERIALS	p lbs. sq. ft.
Concrete on clay	12.5 H
Masonry on clay	25 H
Concrete on shale rock	40 H
Concrete on fine sand	50 H
Concrete on coarse sand or gravel	62 H



D. Molitor, Jan. 1930.

TABLE 46A · HORIZONTAL COMPONENT OF LATERAL PRESSURE = $E = \frac{w}{2}H^2 \tan^2(45° - \frac{\varrho}{2}) = KH^2$.

FILL MATERIAL	WEIGHT per cu.ft. LBS.	Angle of repose ϱ	ANGLE OF FRICTION ϕ			ACTIVE. $\tan^2(45° - \frac{\varrho}{2}) = \theta$	$K = \frac{w}{2}\theta$ LBS	$\tan\phi$ for Concrete.	PASSIVE $\tan^2(45° + \frac{\varrho}{2}) = \theta_p$
			Concrete Surface	Steel Plate	Smooth Wood				
Common Earth, dry	90	30°	24°	–	–	0.333	15.0	0.445	3.00
" " moist	105	40	30	–	–	0.217	11.4	0.577	4.59
" " wet	115	20	15	–	–	0.490	28.2	0.268	2.00
Clay, dry	100	30	24	–	–	0.333	16.7	0.445	3.00
" moist	110	45	30	–	–	0.172	9.4	0.577	5.82
" wet	115	20	18	–	–	0.490	28.2	0.325	2.00
Gravel, clean 1"	104	48	30	–	–	0.149	7.7	0.577	6.80
Sand, gravel, clay	130	26	20	–	–	0.390	25.3	0.364	2.56
Sand, fine dry	100	31	22	15	–	0.320	16.0	0.404	3.10
" " moist	110	40	30	18	–	0.217	11.9	0.577	4.59
" " wet	130	29	10	–	–	0.347	22.6	0.176	2.88
Cr. limestone, fine	100	35	30	20	–	0.271	13.6	0.577	3.70
" " " Coarse	95	45	30	20	–	0.171	8.1	0.577	5.82
Broken Stone	110	60	30	–	–	0.072	4.0	0.577	13.95
Coal, Anth.	55	27	24	16	–	0.376	10.3	0.445	
" Bitum.	50	35	30	18	–	0.271	6.8	0.577	
Coke.	28	45	40	25	–	0.171	2.4	0.839	
Cinders	50	38	30	30	–	0.240	6.0	0.577	
Ashes (soft coal)	42	40	30	–	–	0.217	4.6	0.577	
Charcoal	12	35	24	–	–	0.271	1.6	0.445	
Sawdust	10	30	20	–	20	0.333	1.7	0.364	
Hem. Iron ore	165	35	30	–	–	0.271	22.4	0.577	
Pig Iron, mach. cast	250	50	–	–	–	0.133	16.6	–	
Cement	88	40	30	–	–	0.217	9.6	0.577	
Salt cake	65	45	30	–	–	0.171	5.6	0.577	
Soda ash	75	35	24	–	–	0.271	10.2	0.445	
Cullet, pea size	90	40	30	–	–	0.217	9.8	0.577	
Malt	33	22	24			0.455	7.5	0.445	
Wheat	49	25	24	22	20	0.406	9.9	0.445	
Barley	39	27	24	20	18	0.376	7.4	0.445	
Oats	28	28	25	22	20	0.360	5.0	0.466	
Corn	44	28	23	20	17	0.366	8.1	0.424	
Peas	50	25	16	14	15	0.406	10.2	0.287	
Beans	46	32	24	20	18	0.317	7.3	0.445	
Flaxseed	41	24	22	18	17	0.415	8.5	0.404	
Water	62.4	0	0	–	–	1.000	31.2	0.0	

Art. 47 Problem 47-A. Design a reinforced concrete contilever wall with H = 17.5 ft., and surcharge h = 2.5 ft. Case (g) shown in Fig. 46-A. The analysis follows the description in Art. 46, and illustrates all minor details. The approximate width D of the base slab is found from the formula

$$D = 0.85\,(H+h)\sqrt{\theta} = 0.495\,(H+h) = 10.0 \text{ ft.} \qquad \text{47-A.}$$

The weight Q is the resultant of the several partial weights $P + \acute{g} + \acute{g}' + g''$, and its position is found by taking moments about the heel.

The earth pressure E is now found from Eq. 46-A as 6570# with a lever arm r, from Eq. 46-B, and the resultant earth pressure E′ is found graphically, as indicated in Fig. 47-A. The resultant R is also found graphically by combining E′ with Q. The vertical component N of R then gives the soil pressures by formula 46-C, all as given on the drawing.

The table of stem moments is figured for heights of 4 to 26 ft. from the formula

$$M = Er = KH\left(Hh + \tfrac{3}{8}H^2\right) \qquad \text{47-B.}$$

wherein H is treated as a variable, while h = 2.5 ft. constant.

The required steel areas for different heights of stem are thus found and may be used for other examples involving similar data. The unit stresses used are $f_c = 650$ lbs. sq. in. and $f_s = 16000$ lbs. sq. in.

Problem 47-B is given to illustrate a case of cantilever wall with high surcharge using the formulas of Fig. 46-A, case (h). For the data given, the width of base slab is found from Eq. 47-A as $D = 0.85(H+h)\sqrt{\theta} = 9.96$ ft., used 9.5 ft.

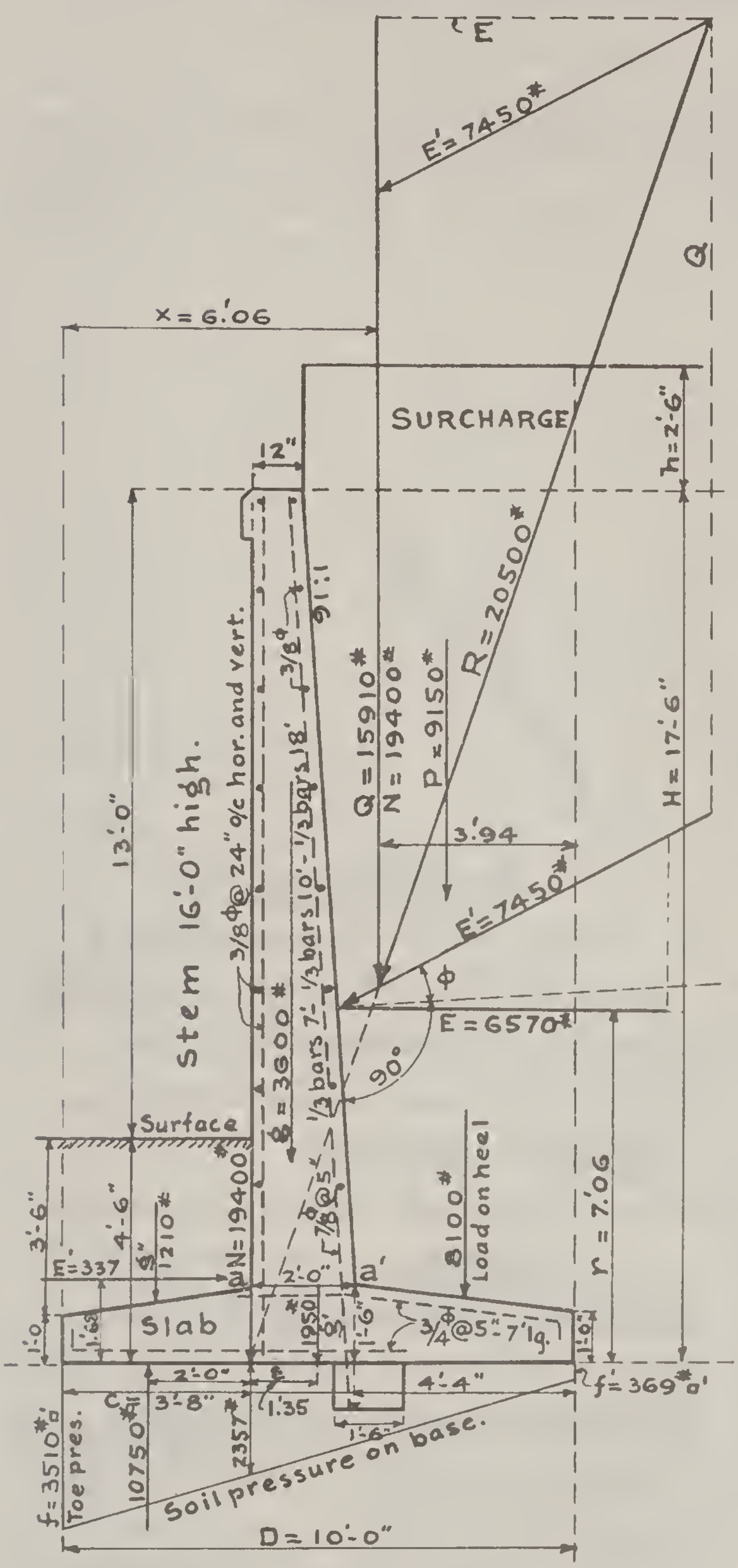
The angle $\psi = 45° + \varsigma/2 = 61°50'$, determines the surface of rupture which is required to find the height h of equivalent surcharge.

The toe pressure is found as $f = \frac{2 \times 12200}{3 \times 2.95} = 2760$ lbs. per sq. ft. because $c < D/3$.

The wall would not be safe against sliding without the key wall and the earth in front of the toe.

Other features of the design are given on the drawing and require no further description.

PROBLEM 47A. REINFORCED CONCRETE CANTILEVER WALL.



DATA FOR DESIGN.

$f_c = 650$ lbs., $f_s = 16000$ lbs., $A_s = \dfrac{M \text{ ft.lbs.}}{1165(d-3'')}$

$w = 100$ lbs., $\varrho = 30°$, $\phi = 24°$, Soil pres. = 3500#□'.

$\tan^2(45° - \varrho/2) = 0.333$., $K = \frac{100}{2} \times 0.333 = 16.66$

$E = \frac{w}{2}(H+2h)H\tan^2(45° - \varrho/2) = KH(H+2h)$ .... (1)

$r = \dfrac{Hh + 3/8 H^2}{H+2h}$ .... (2)

$M = rE = KH(Hh + 3/8 H^2) = 41.7H^2 + 6.25H^3$ for $h = 2.5$ – (3)

see table below for steel required in stem.

$E = 16.66(17.5 + 2\times 2.5)17.5 = 6570$ lbs.

$r = \dfrac{17.5 \times 2.5 + 0.375 \times \overline{17.5}^2}{17.5 + 2 \times 2.5} = 7.06$ ft.

$D = 0.85(H+h)\sqrt{\tan^2(45 - \varrho/2)} = 0.495(H+h) = 10'0''$

$P = 18.6 \times 4.95 \times 100 = 9150$ lbs $\times 2'.5 = 22,850$ ft.lbs.

$g = 16 \times 1.5 \times 150\# \times 3600$ " $\times 5.6 = 20,150$ " "

$g' = (1.5\times 2 + 8 \times 1.25)150 = 1950$ " $\times 5.0 = 9,750$ " "

$g'' = 3.67 \times 3.3 \times 100 = 1210$ " $\times 8.25 = 10,000$ " "

Total Vert. Load $Q = 15910$ lbs $\times 3.94 = 62,750$ ft.lbs.

The graphic composition of forces gives R, with the vertical component N = 19400 lbs. acting on the base at a point $\varepsilon = 1.35$ from the center, where R prolonged cuts the base. Then the soil pressure $f = \frac{N}{D}\left(1 \pm \frac{6\varepsilon}{D}\right) = \frac{19400}{10}\left(1 + \frac{6\times 1.35}{10}\right) = 3510\#□'$ --- (4)

and $f' = 369\#□'$

Reinforcing steel

The stem $M_a = E\ (r - 1.5) = 6570 \times 5.56 = 36500$ ft.lbs. $A_s = 1.49$

Heel $M'_a = 8100 \times 2.16 = 17500$ ft.lbs. $A_s = 1.00 = 3/4^\phi @ 5''$

Toe $M_a = 10750 \times 2 - 1210 \times 1.83 = 19280$ ft.lb., $A_s = 1.11□''$

Safety against sliding on the base, $N\tan\phi > E$.

$N\tan\phi = 19400 \times 0.445 = 8650\#$, Safety $= \frac{8650}{6570} = 1.32$

Use 12"×18" key on base for additional safety.

The factor of safety against overturning $= \gamma = \frac{Nx}{Er} = 2.53$, at the same time that the toe pressure f is not excessive.

Total Concrete area = 37.0 sq.ft., 150 lbs. of bars per lin.ft.

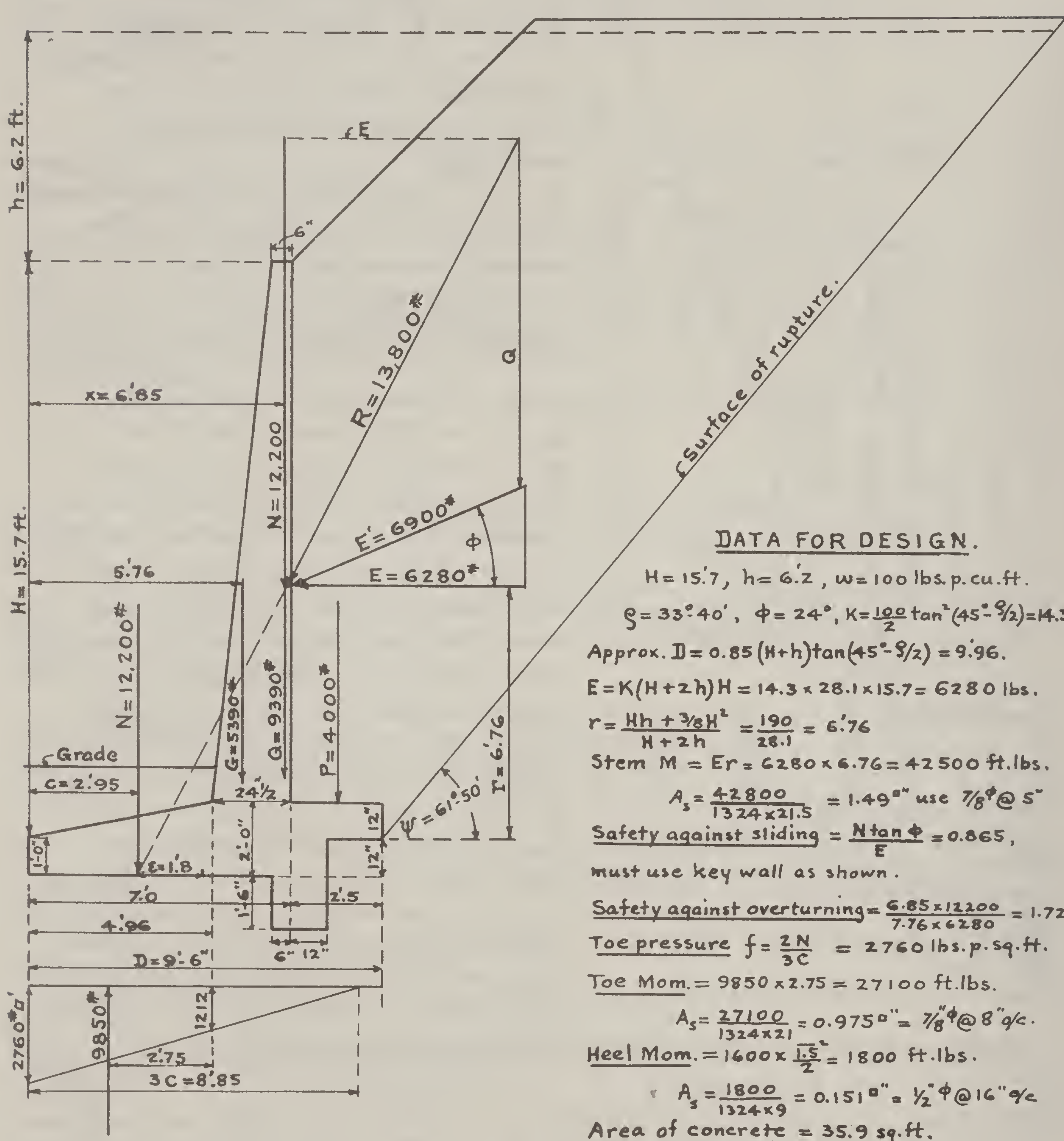
STEM MOMENTS FOR VARIOUS HEIGHTS H AND h = 2.5 ft. AS PER EQ. 3

H ft.	M ft.lbs	d in	A_s Steel area sq.in.	H ft.	M ft.lbs	d in	A_s Steel area sq.in	H ft.	M ft.lbs	d in.	A_s Steel area sq.in.
4	1055	15	0.076	12	16700	21	0.798	20	66,660	27	2.38
6	2850	16.5	0.181	14	25290	22.5	1.11	22	86,680	28.5	2.91
8	5866	18	0.336	16	36250	24	1.48	24	110,500	30	3.51
10	10420	19.5	0.544	18	49.900	25.5	1.90	26	138,000	31.5	4.16

All values for 1 ft. length of wall.

D. Molitor
Jan, 1930.

PROBLEM 47B. REINFORCED CONCRETE RETAINING WALL.



DATA FOR DESIGN.

$H = 15.'7$, $h = 6.'2$, $w = 100$ lbs. p. cu. ft.

$\varrho = 33°40'$, $\phi = 24°$, $K = \frac{100}{2}\tan^2(45° - \varrho/2) = 14.3$

Approx. $D = 0.85(H+h)\tan(45° - \varrho/2) = 9.'96$.

$E = K(H+2h)H = 14.3 \times 28.1 \times 15.7 = 6280$ lbs.

$r = \frac{Hh + 3/8 H^2}{H + 2h} = \frac{190}{28.1} = 6.'76$

Stem $M = Er = 6280 \times 6.76 = 42500$ ft. lbs.

$A_s = \frac{42800}{1324 \times 21.5} = 1.49 \square''$ use $7/8\phi$ @ 5"

Safety against sliding $= \frac{N \tan\phi}{E} = 0.865$,

must use key wall as shown.

Safety against overturning $= \frac{6.85 \times 12200}{7.76 \times 6280} = 1.72$

Toe pressure $f = \frac{2N}{3C} = 2760$ lbs. p. sq. ft.

Toe Mom. $= 9850 \times 2.75 = 27100$ ft. lbs.

$A_s = \frac{27100}{1324 \times 21} = 0.975 \square'' = 7/8''\phi$ @ 8" o/c.

Heel Mom. $= 1600 \times \frac{\overline{1.5}^2}{2} = 1800$ ft. lbs.

$A_s = \frac{1800}{1324 \times 9} = 0.151 \square'' = 1/2''\phi$ @ 16" o/c

Area of concrete $= 35.9$ sq. ft.

D. Molitor

Problem 47-C Safety of walls against Sliding. The question of safety against sliding on the base, previously referred to in Art. 46, may not be the only manner of failure by sliding, as there may be planes of weakness below the wall base for which the stability condition is one of balance between the active earth pressure E_1 below the footing, against an equal or greater passive earth resistance E_p in front of the wall as shown on Fig. 47-C.

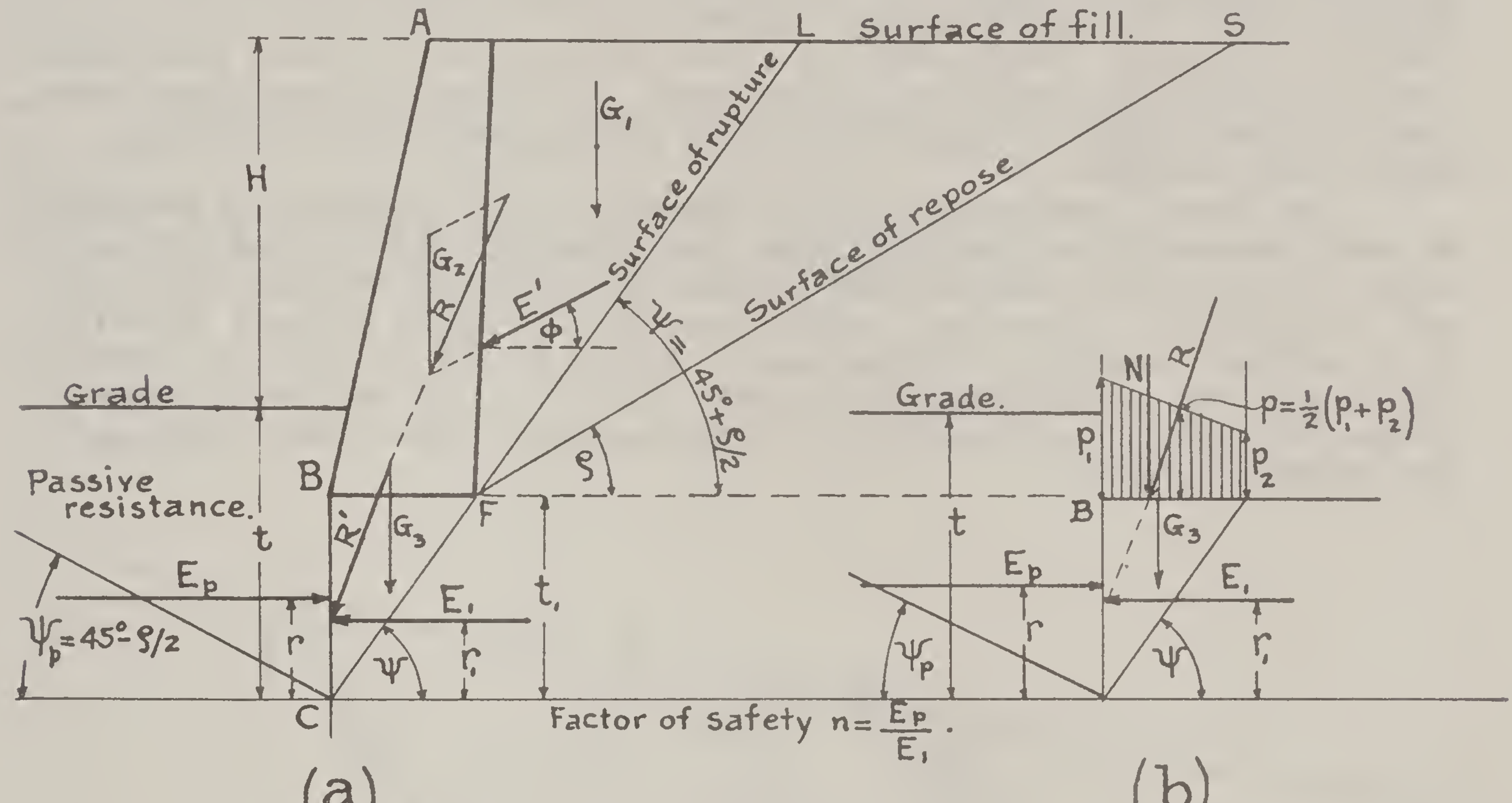


Fig. 47 C.

The wall AB supported at BF is subjected to an earth pressure E' with a resultant R on the soil, producing unit soil pressures p_1 and p_2, the average of which is p, see Fig. b. This unit load p is taken as a surcharge on the pressure wedge BFC from which the active earth pressure E_1 for the height t_1 is evaluated. The passive resistance E_p to the left of the wall and acting on the plane BC, is found for the height t, and must exceed E_1, with a certain factor of safety $n = E_p/E_1$.

Eq. 46-A furnishes a value for E_1, and E_p may be evaluated from Eq. 43-B, resulting in the following formulas:

$$E_1 = \frac{w}{2} t_1 (t_1 + 2p) \tan^2(45° - \varrho/2) \quad \text{and} \quad r_1 = \frac{pt_1 + 3/8\, t_1^2}{2p + t_1} \quad \text{.....} \; 47\text{-C.}$$

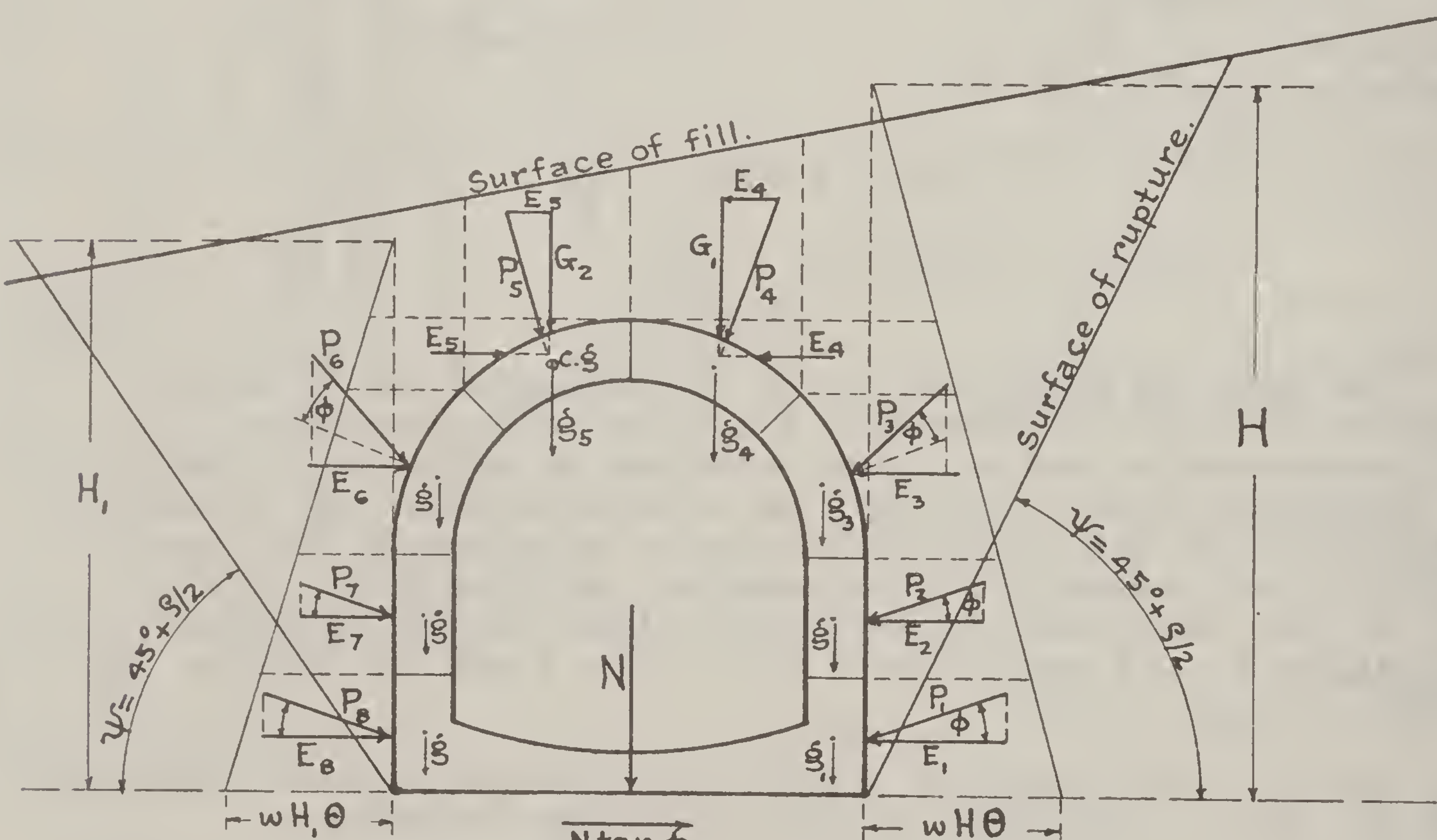
$$E_p = \frac{w}{2} t^2 \tan^2(45° + \varrho/2) \text{ acting at } r = 3/8\, t \text{ above C} \quad \text{.......} \; 47\text{-D.}$$

Art. 48 Arches under ~~an~~ Embankments.

The lateral earth pressures acting on a culvert or underpass roadway may be estimated according to the Rankine-Coulomb theory although these pressures may be vastly altered by the method employed in placing the back-fill against and over the finished structure.

Thus it may happen that the lateral pressure attains even greater intensity than that due to natural earth pressure, if the backfill is excessively tamped. Or, when little or no tamping is done, there may be a deficiency of lateral pressure and the structure may be called upon to carry the vertical top load without the assistance of the side pressure to develop full arch action.

The total horizontal force due to lateral earth pressure on one side may be greater than the total force on the opposite side, a condition encountered when the surface of the fill is not horizontal. However, such inequality would adjust itself by a certain lateral displacement which would result in developing a passive resistance at least equal to the active lateral pressure, less the frictional resistance of the structure on its basal support.



Condition $E_8 + E_7 + E_6 + E_5 + N\tan\phi = E_4 + E_3 + E_2 + E_1$

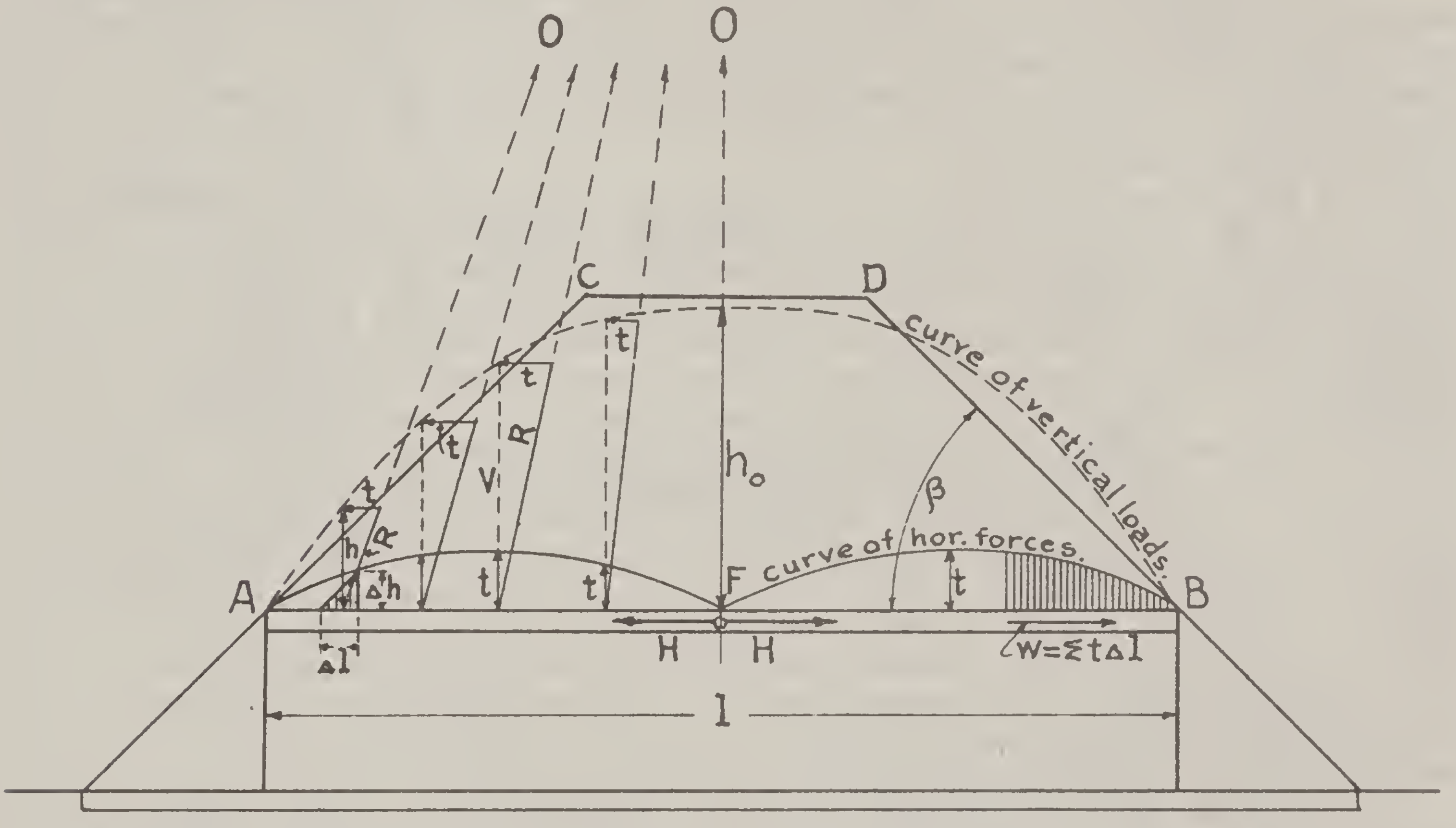
Fig. 48 A

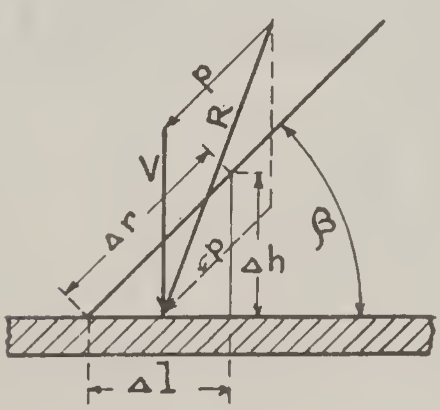
Assuming normal earth pressures on both sides, and the weight of the earth over the top of the structure, Fig. 48-A illustrates the method of evaluating the earth pressures and loads P acting on the various voussoir sections. The condition equation for the horizontal forces must be satisfied, observing that the value $N \tan\phi$ should be the absolute minimum. If necessary the values E_8, E_7, E_6, E_5 may be increased to produce the balance between the horizontal forces.

With the forces P established, the line of thrust may be constructed and the arch dimensions arrived at in the usual manner.

<u>The longitudinal force on a Culvert</u> is difficult to evaluate with any degree of accuracy. The following is to be regarded as a mere approximation.

Let Fig. 48-B represent a longitudinal section of the structure taken through the fill ACDB.





$V = \Delta r\, w h \cos.\beta$.
$p = \Delta h\, w h \cos.\beta\,(1 - \sin\beta \tan\beta)$.
R = resultant of V and p.
W = the shaded area with ordinates t, and is maximum at F where the whole area from B to F represents the horizontal force H.

Fig. 48 B.

The analysis is based on finding the resultant R for a length Δl near the toe of the fill and this resultant will intersect the axis of the dam in O, which becomes the common pole for all resultants R. The horizontal components of the several forces R are then plotted as ordinantes to the base $\overline{AB}$, and the area of this curve, on one side of the center F, represents the total horizontal force H.

This method is given by Dr. H. Krey (Erddruck, Erdwiderstand) p. 170, with the reminder that it cannot be theoretically substantiated though it may serve a useful purpose in the absence of a better method.

To analyse a given arch ring by line of thrust method, when the externally applied forces have been determined as previously described. Referring to Fig. 48C, proceed as follows:

I. Divide the arch ring into a suitable number of voussoirs, or hypothetical arch stones, so as to insure a joint at the crown and one at each support. The location of the voussoirs is a matter of convenience. Then find the external forces acting on the several voussoirs and combine these with the vertical weights g to obtain the total forces p_1 to p_8, to be carried by the arch ring.

II. Construct a force polygon by adding graphically the several forces p_1 to p_8 in their proper order, true directions and magnitudes, using any convenient scale of forces. The resultants P_1 and P_2, of the two groups of applied forces to the left and right of the crown joint c, are thus found in direction and magnitude.

A point on the line of action of each of the two forces, as d_1 for P_1 and d_2 for P_2, is found by drawing any trial equilibrium polygon through the several forces p_1 to p_8 for any convenient pole O′, with pole distance t. The points d_1 and d_2 are the intersections of the closing rays r_1 and r_2 with t, and fix the positions of the resultants P_1 and P_2. This much of the drawing may be made in ink so as to permit of erasing subsequent construction lines drawn in pencil for the purpose of locating the most probable line of thrust.

III. An equilibrium polygon can now be drawn through any three assumed points c, a and b of the crown and springing joints, and as a first trial it is well to choose the axial points of these three joints as giving the best indication of such shifts as may become desirable for subsequent trials. The line of thrust corresponding to any equilibrium polygon is merely another broken line joining the points of intersection of the resultant rays with the voussoir joints.

To draw an equilibrium polygon through the three axial points a-c-b, it is necessary to first find the crown thrust T acting at c. This thrust, is horizontal for symmetrical load-

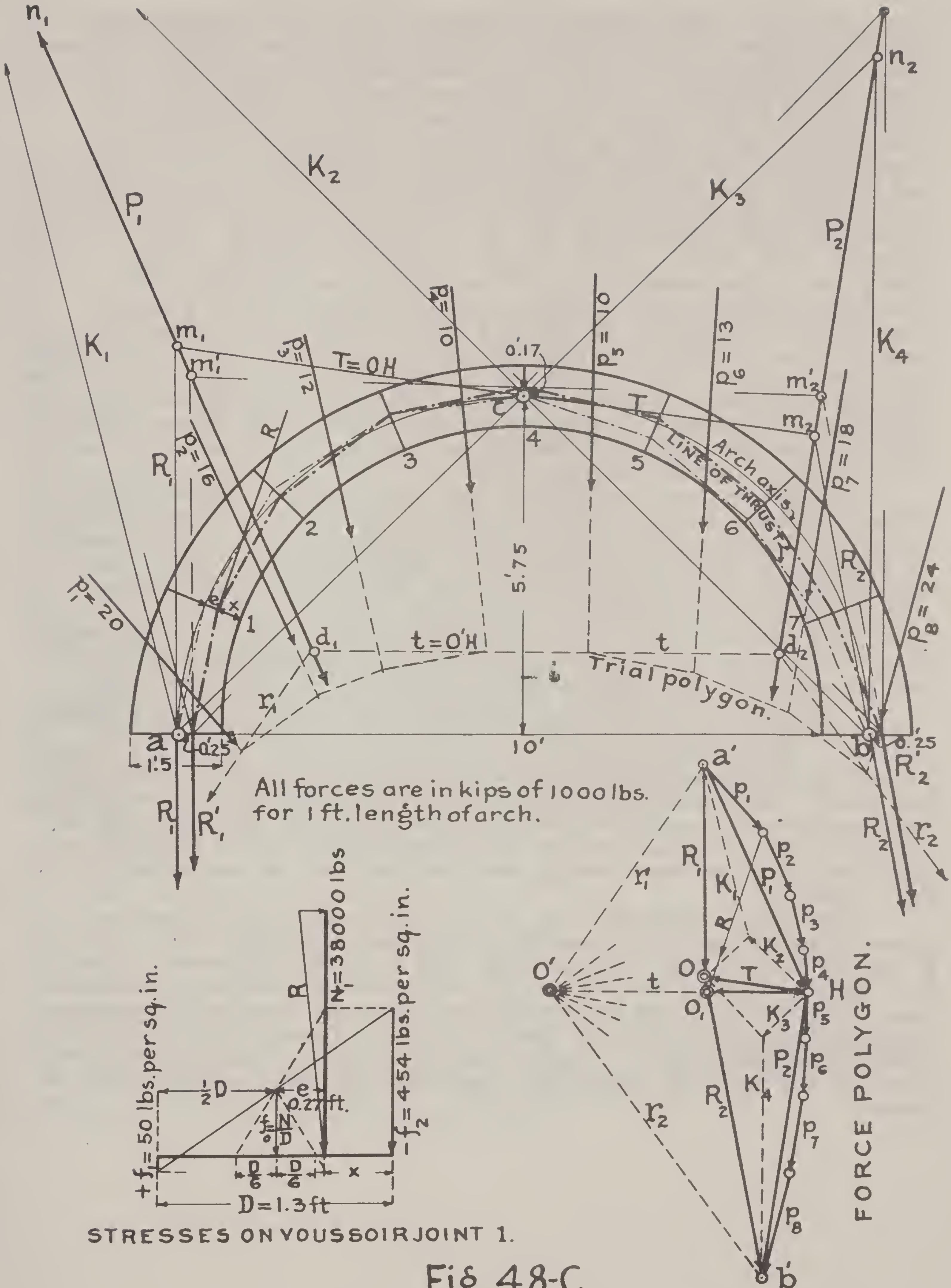

All forces are in kips of 1000 lbs. for 1 ft. length of arch.
Arch axis
LINE OF THRUST
Trial polygon.
T = O H
t = O'H
5.75'
10'
0.17
1.5
0.25
N = 38000 lbs
+f1 = 50 lbs. per sq. in.
-f2 = 454 lbs. per sq. in.
D = 1.3 ft
0.27 ft.
FORCE POLYGON.
STRESSES ON VOUSSOIR JOINT 1.

Fig. 48-C.

ing, while for unsymmetrical loading, as in the present case, its direction is found in the force polygon as follows: Since the points a-c-b, if they are to be on the line of thrust, must necessarily be points of zero moments, therefore, the reactions of the force P_1 on the half arch $\overline{ac}$ must be K_2 (through b and c) and K_1 (through a and n_1). Likewise the reactions of the force P_2 on the half arch $\overline{bc}$, must be K_3 and K_4. But the crown thrust T is the resultant of K_2 and K_3 and may be found in the force polygon, by resolving P_1 into K_1 and K_2, also P_2 into K_3 and K_4 and then finding the resultant of K_2 and K_3 equal to $T = \overline{OH}$. By drawing T through the point c and parallel to $\overline{OH}$ in the force polygon, the points m_1 and m_2 are located, and the reactions R_1 and R_2 are fully determined. Finally with O as the pole, draw the equilibrium polygon through the axial points a-c-b, (not shown) and then connect the points of intersection of this equilibrium polygon with the several voussoir joints, to obtain the line of thrust as shown by a light dotted line. In like manner other lines of thrust may be constructed through any three points of the crown and springing joints.

By inspection it is seen that the line of thrust just found, passes close to the intrados at joint 6, which appears to be a critical section, while in the left half of the arch it closely follows the arch axis. This indicates that the true line of thrust must come closer to the arch axis over the right half of the arch, while on the left half the new line must depart more from the axis in order that the internal work of deformation may be reduced to a minimum, and thus more nearly equalize the stresses at all points of the arch ring.

For these reasons we construct a new equilibrium polygon and resulting line of thrust, choosing the three points as follows: Move the points a and b to the right, or middle third points of these sections, and move the point c up 0.17 ft. to the outer third point. By repeating the above method we locate the new pole O_1, and the heavy dotted line as the new line of thrust. The stresses on any arch section as joint 1, are found graphically as shown on Fig. 48C, or they may be computed by Eqs. 46-C.

Now that a new line of thrust has been found for the given case of loading, revealing critical sections at a, 1, c, 6 and b, how is one to know that the critical stresses found for these sections are actually the minimum obtainable values? The criteria will now be given for locating the most probable line of thrust, such that the critical stresses simultaneously attain minimum values. This requires finding the most probable line of thrust from among all the possible ones which might be drawn.

Criteria for locating the most probable line of thrust have been proposed by Gerstner 1831, Moseley 1837, Hagen 1844, Bauernfeind 1846, Schwedler 1859, Culmann 1866, Durand-Claye 1867, Winkler 1867, Heinzerling 1869, Von Ott 1870, Belpaire 1877, Müller-Breslau 1886, J. Weyrauch 1897, Landsberg 1901.

According to Menabrea's Law of least work, the statement regarding the location of the most probable line of thrust in an arch ring, would be that it must correspond to a minimum work of deformation. The actual internal work due to bending moments $\underline{M}$ and direct stresses $\underline{N}$, by Eq. 15K, p. 42, Kinetic Theory of Eng. Structures, is

$$W=\frac{1}{2EI}\int M^2dx+\frac{1}{2EF}\int N^2dx \text{ - } 48A$$

Since $\underline{M} = \underline{N}\underline{e}$, where $\underline{e}$ is the eccentricity of $\underline{N}$, measured from the arch axis, therefore Eq. 48A may be written

$$W=\frac{1}{2EI}\int N^2e^2dx+\frac{1}{2EF}\int N^2dx \text{ - - - - - - - - - - - - - - - - - - - } 48B$$

Referring to the force polygon Fig. 48C, it will be seen that the resultants $\underline{R}$, and hence their normal components $\underline{N}$, acting on the several voussoir sections, are only slightly affected by small shifts in the pole 0, hence the second term in Eq. 48B may be regarded as practically constant so long as the arch ring and its loading remain unchanged. Also, the products $\underline{N}^2\ \underline{e}^2$ in the moment function will depend essentially on the variable $\underline{e}^2$. Therefore, the internal work due to moments will become minimum when $\int \underline{e}^2\underline{dx}$ is minimum, and since we are not interested in the numerical value of W but only in the conditions which reduce its value to a minimum, therefore, the second term disappears as a governing condition.

Hence, the condition which reduces the actual work of deformation to a minimum is that $\int e^2dx$ = min., and if the voussoirs are spaced at equal intervals making dx constant, then the condition is simply that Σe^2 = min. Stated in words, <u>the most probable line of thrust is the one which reduces to a minimum, the residual departure from the arch axis (center line) according to the method of least squares.</u>

For example, if a line of thrust can be made to coincide with the arch center line, then this thrust line will be the most probable one of all the possible lines for the given case of loads, as the value Σe^2 will then be zero. It was for this reason that the first equilibrium polygon was passed through the three axial points $\underline{a}$-$\underline{c}$-$\underline{b}$ in Fig. 48C, so as to reveal the character of our problem and then decide what shifts to make in order to comply with the above criterion. Thus, for the line of thrust drawn through the three axial points, with values of $\underline{e}$ in inches, we obtain $\Sigma e^2 = 85.67$, while for the heavy line of thrust through the middle third points $\Sigma e^2 = 40.72$. A line of thrust drawn through $\underline{a}$ and the outer third points at $\underline{c}$ and $\underline{b}$ (not shown) gives $\Sigma e^2 = 26.4$ which may be taken as sufficiently close for a minimum value.

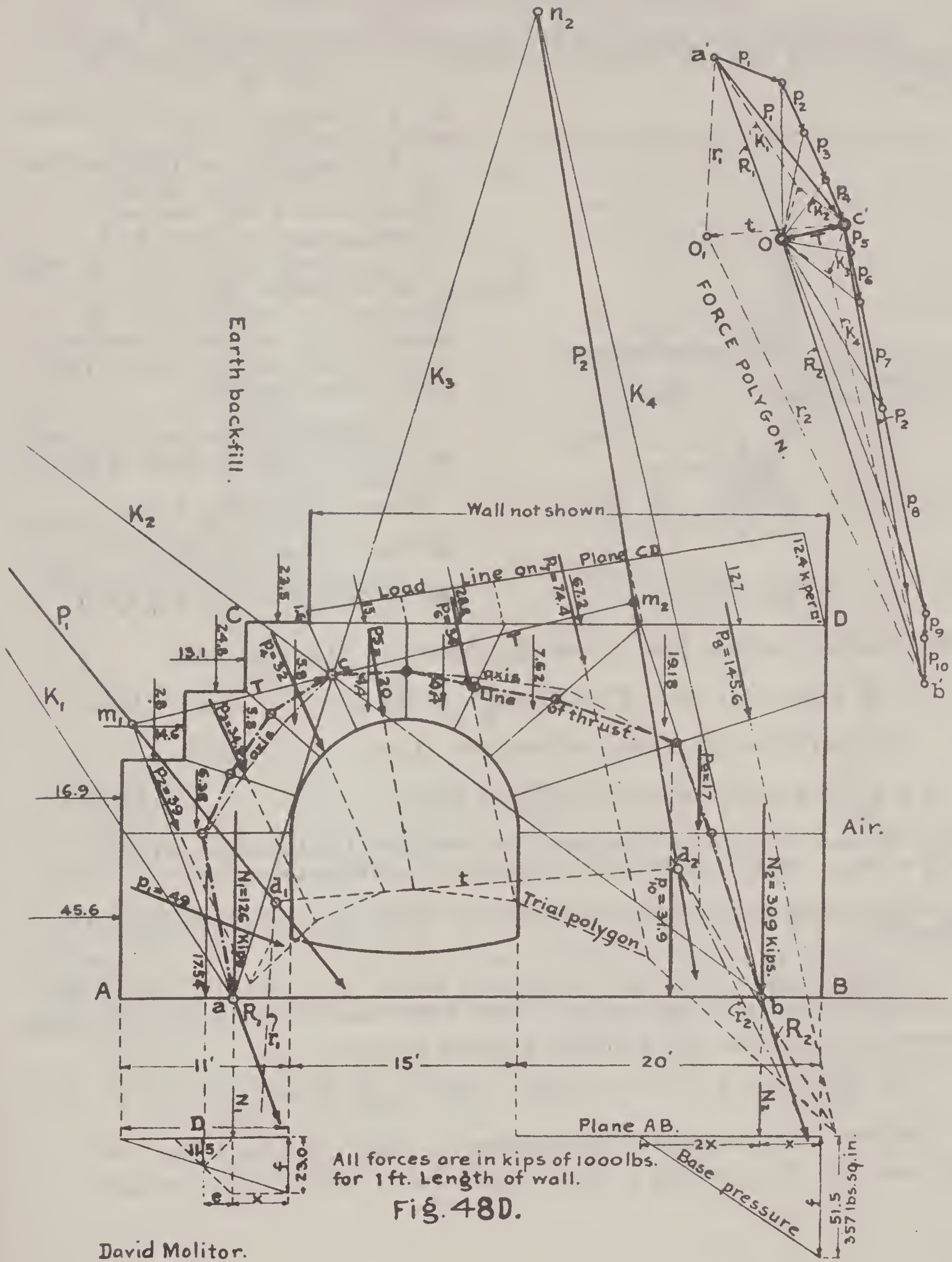
There is a further condition which must be satisfied in order to balance the positive and negative values of e. It is that the line of thrust (or equilibrium polygon) must intersect the arch axis in at least three points, and when the arch and its loading are symmetrical there must be at least four such points of intersection. Our final line of thrust has three axial intersections and complies with this requirement.

The method here outlined for the analysis of arches should always be employed when dealing with short span, semi-circular or high arches, having rather thick rings and subjected to non-parallel loads, as for arches under high earth fills.

All methods for the evaluation of the three redundant conditions in arches of the kind here treated, are more or less speculative, but yield results which are quite within knowable limits, especially when the heterogeneous character of masonry is considered. Whether we locate the line of thrust by the method of least work, by the elastic theory, or by a few trials, observing the criteria above stated, makes very little difference except in labor involved, and certainly the least laborious solution is the most acceptable.

The above method was devised by the writer for the analysis of the Gatun Lock walls of the Panama Canal in 1907, as illustrated in Fig. 48D. The lettering corresponds to that employed in Fig. 48C, so that the description need not be repeated.

When dealing with large flat arch ribs, or barrel arches with thin concentric rings, carrying vertical loads, then the theory of elasticity affords the best method of stress analysis. See the writers Chap. 15, Kinetic Theory of Eng. Structures, for an exhaustive treatise on this subject.



All forces are in kips of 1000 lbs.
for 1 ft. Length of wall.

Fig. 48D.

David Molitor.
1907.

Art. 49. Sheet piles in level earth with horizontal force P, acting at the top as shown Fig. 49-A. All loads per 1 ft. of piling.

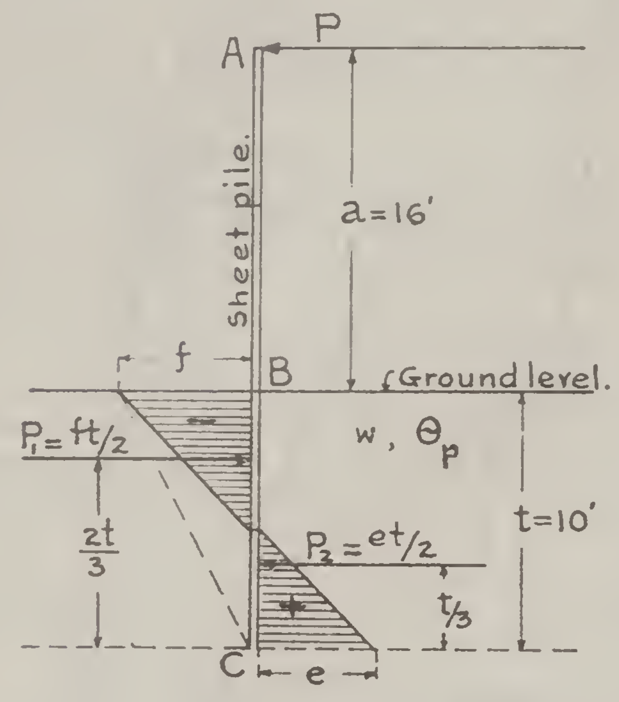


Fig. 49A.

Let w = weight of earth, pounds per cu. ft.

f = passive resistance per sq. ft. at ground level, which should not exceed $\frac{1}{2} wt\theta_p = \frac{1}{2} wt \tan^2(45° + \varphi/2)$ for a factor of safety of two.

e = pounds per sq. ft. pressure at the bottom of sheet pile, which likewise should not exceed the allowable value $\frac{1}{2} wt\theta_p$.

Two condition equations may be written as follows: $\Sigma P = 0$ and $\Sigma M_c = 0$.

Hence from Fig. 49-A,

$$\Sigma P = P - P_1 + P_2 = 0$$

$$\Sigma M_c = P(a+t) - \frac{2t}{3}P_1 + \frac{t}{3}P_2 = 0.$$

Substituting values for P_1 and P_2 from the figure, then

$$P - \frac{ft}{2} + \frac{et}{2} = 0 \quad \text{and} \quad P(a+t) - \frac{t^2 f}{3} + \frac{t^2 e}{3} = 0 \ \ldots\ldots\ldots\ldots\ 49\text{-A}.$$

The solution of these equations gives

$$f = \frac{2P}{t^2}(3a+2t) \quad \text{and} \quad e = \frac{2P}{t^2}(3a+t) \ \text{-------------}\ 49\text{-B}.$$

which values should not exceed the passive resistance $wt\theta_p$, or $\frac{1}{2} wt\theta_p$ when allowing a factor of safety two.

The maximum bending in the sheet pile occurs at about 0.2t below B.

EXAMPLE 49-A. For wet sand with w=130 lbs. per cu. ft., and $\varphi = 29°$, $\theta_p = \tan^2(45° + \frac{29}{2}) = 2.88$, then with sheet piling 26 ft. long, driven 10 ft. into the ground, Eqs. 49-B give

$$f = \frac{2P}{100}(3\times16 + 2\times10) = 1.36P, \text{ and } e = \frac{2P}{100}(3\times16 + 10) = 1.16P.$$

These values should not exceed $\frac{1}{2} wt\theta_p = \frac{130}{2} \times 10 \times 2.88 = 1862$ lbs. per sq. ft., hence a safe value for P should not exceed

$f/1.36 = \frac{1862}{1.36} = 1370$ pounds. The maximum bending moment on the pile, per ft. of wall, will be about $P(a+0.2t) = 1370 \times 18 = 24{,}600$ ft. lbs. to develop the available resistance for the 10 ft. penetration with a safety of two.

Art. 50. Sheet piling back filled with earth on one side, and a uniform live load q per sq. ft. on the fill.

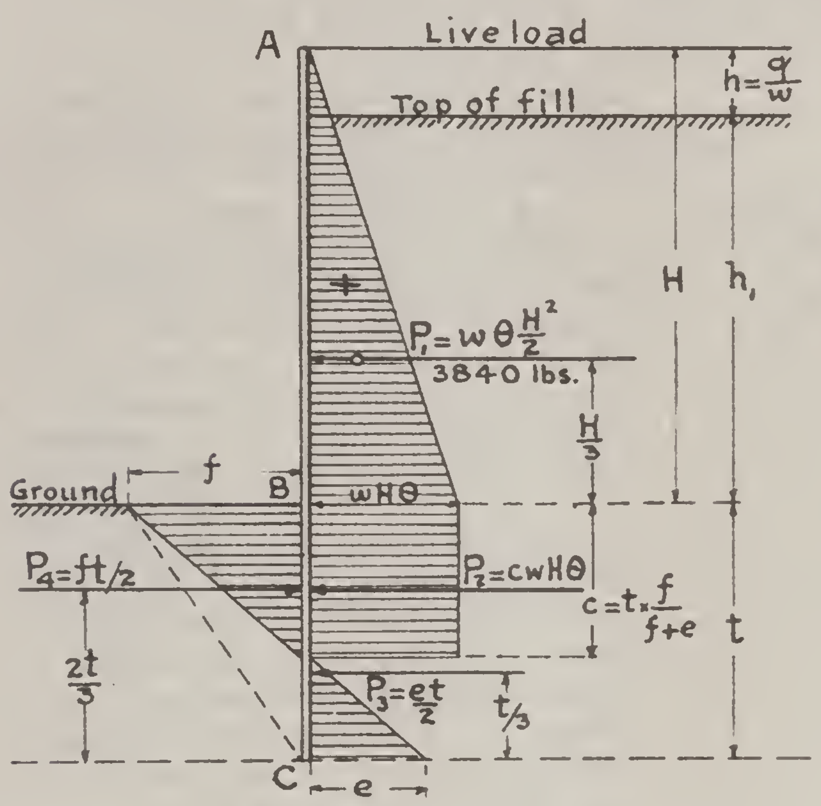


Fig. 50A.

Let w = wgt. of earth, lbs. per cu. ft.

$\theta = \tan^2(45° - \varphi/2)$.

P_1, P_2, P_3 and P_4 are earth pressures as indicated in Fig. 50-A.

f = passive resistance, lbs. per sq. ft. at ground level which should not exceed $\frac{1}{2} wt\theta_p$ for $\theta_p = \tan^2(45° + \varphi/2)$.

e = passive resistance lbs. per sq. ft. at bottom of sheet pile.

The condition equations for $\Sigma P = 0$, and $\Sigma M_c = 0$ may then be written as follows:

$$\left.\begin{aligned} &P_1 + P_2 + P_3 - P_4 = 0, \text{ or } \frac{w\theta H^2}{2} + cwH\theta + \frac{et}{2} - \frac{ft}{2} = 0 \\ \text{and } &\frac{w\theta H^2}{2}\left(\frac{H}{3} + t\right) + cwH\theta\left(t - \frac{c}{2}\right) + \frac{et^2}{6} - \frac{ft^2}{3} = 0 \end{aligned}\right\} \text{------ 50-A.}$$

where $c = t\frac{f}{f+e} = 0.60\,t$ approximately.

Equations 50-A, with numerical values for any given problem, will serve to find the unit soil pressures f and e for an assumed value of t. These soil pressures should not exceed the passive resistance $\frac{1}{2} wt\theta_p$ for a factor of safety of two.

The point of max. moment in the sheet piling is at about $t/3$ below B.

Substituting the approx. value of $c = 0.6t$ into Eqs. 50-A, and solving for f and e, gives

$$\left.\begin{aligned} f &= \frac{w\theta H^2}{t^2}(H + 2t) + 1.32\,wH\theta \\ e &= \frac{w\theta H^2}{t^2}(H + t) + 0.12\,wH\theta \end{aligned}\right\} \text{--------------- 50-B.}$$

Example 50-A. $H = 16$ ft., $t = 10$ ft., $w = 90\#$, $\varrho = 30°$, $\theta = 0.333$, $\theta_p = 3.00$, then Eqs. 50-B give $f = 3398$ lbs. sq. ft. and $e = 2054$ lbs. sq. ft. The passive resistance is only $wt\theta_p = 2700$ lbs., without any allowance for safety. This means that t must be made greater. Max. moment = 8.67 x 3840 = 33,290 ft. lbs.

Art. 51. The effect of Relieving Platforms in reducing the active pressure on the face wall is important and may be briefly illustrated without regard to the manner in which the wall is supported. The questions relating to wall support are taken up separately in dealing with back anchors and sheet pile penetration.

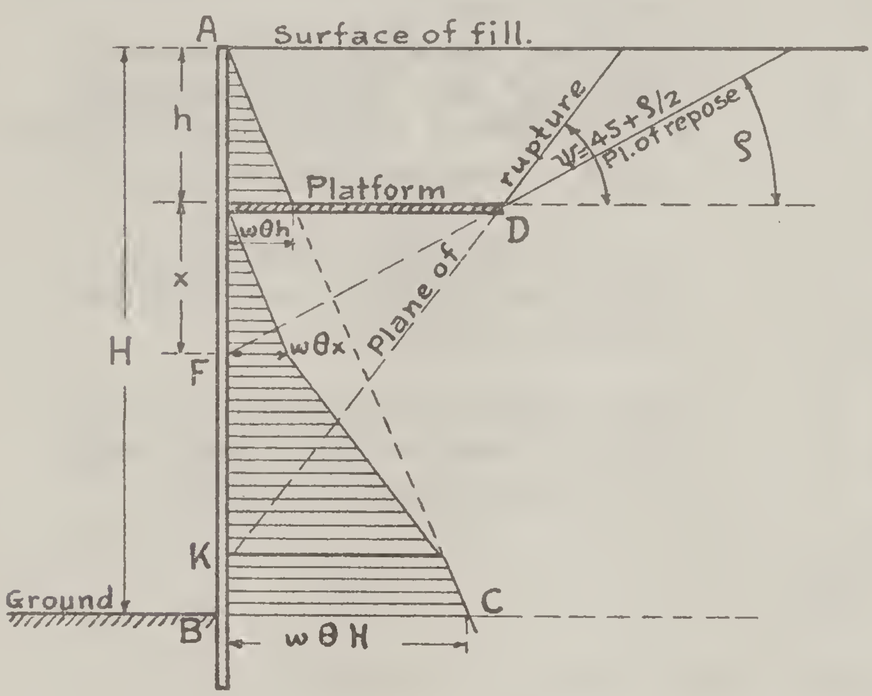
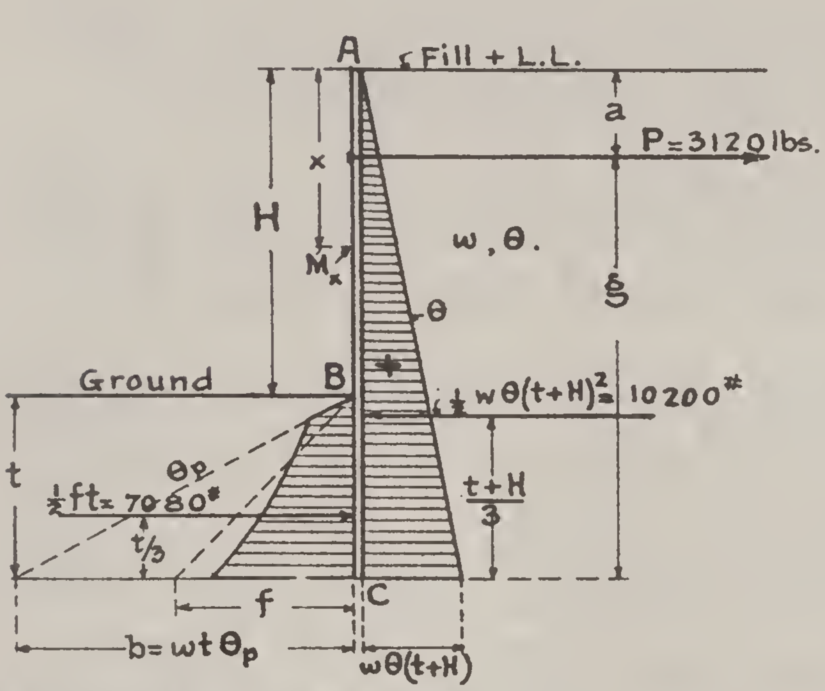


Fig. 51 A.

The platform may be supported on piles and takes weight off the fill beneath the platform, thus relieving the horizontal pressure against the front wall in a manner indicated by Fig. 51-A.

The pressure area for the height h above the platform has a base $w\theta h$ lbs. per sq. ft. and the same line is extended down to B where the value becomes $w\theta H$ lbs. per sq. ft. Drawing the natural slope line through D to F, and the plane of rupture through D to K determines the points where the pressure changes, giving the shaded area as the total pressure area active against the face wall.

Art. 52. Sheet piling back filled and with back anchor.



$\theta = \tan^2(45° - \varsigma/2)$ - active.

$\theta_p = \tan^2(45° + \varsigma/2)$ - passive.

w = weight of 1 cu. ft. Earth fill.

ς = angle of repose of earth.

Fig. 52 A.

Condition equations.

$$P - \frac{w\theta}{2}(t+H)^2 + \frac{ft}{2} = 0$$

$$M_c = Pg - \frac{w\theta}{6}(t+H)^3 + \frac{ft^2}{6} = 0,$$

to find f and P for an assumed t.

A certain factor of safety should be provided against failure at the bottom so that the passive resistance $b = nf$, may have a factor of safety $n = \frac{wt\theta_p}{f}$, which should be about two.

$$\left.\begin{aligned} P + \frac{ft}{2} &= \frac{w\theta}{2}(t+H)^2 \\ Pg + \frac{ft^2}{6} &= \frac{w\theta}{6}(t+H)^3 \end{aligned}\right\} \text{------ 52-A.}$$

giving the value

$$f = \frac{w\theta(t+H)^2\left(\frac{t+H}{3g} - 1\right)}{t\left(\frac{t}{3g} - 1\right)} \text{ ----- 52-B.}$$

Example 52-A. $H = 16$ ft., $t = 8$ ft. $w = 115$ lbs, $\varsigma = 32°$, $\theta = 0.308$, $\theta_p = 3.24$, $g = 20$ ft., which values give $f = \frac{115 \times 0.308 \times 24^2\left(\frac{24}{60} - 1\right)}{8\left(\frac{8}{60} - 1\right)} = + 1770$ lbs. sq. ft.

The total passive resistance is $b = wt\theta_p = 115 \times 3.24 \times 8 = 2980$ lbs. per sq. ft. which is not quite 2f.

The pull on the anchor $P = \frac{w\theta}{2}(t+H)^2 - \frac{ft}{2} = 3120$ lbs. per ft. of wall.

To find the point of max. moment, write the moment equation for any point x below A and make $\frac{\partial M}{\partial x} = 0$ to find x, thus =

$$M_x = P(x-a) - \frac{w\theta}{6}x^3, \text{ and } \frac{\partial M}{\partial x} = P - \frac{w\theta}{2}x^2 = 0,$$

making $x = \sqrt{\frac{2P}{w\theta}} = 13.27$ ft., and $M_x = P(13.27-4) - 35.42 \times \frac{\overline{13.27}^3}{6} = 15{,}200$ ft. lbs.

Art. 53 The Combined Action of Earth and Water on Sheet Piling.

This problem is usually encountered in designing Quay walls or marginal wharves and involves many uncertainties, the nature and magnitude of which cannot be appraised with accuracy.

Case I. Coarse Material for Back-fill. The physical aspects of this problem will first be discussed by assuming a line of sheet piling with water on one side and rip rap or crushed stone on the other side, both at the same surface level as shown in Fig. 53-A.

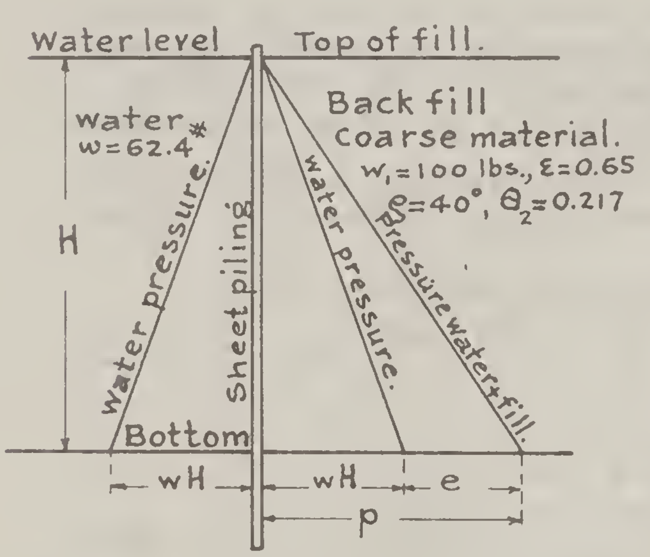


Fig. 53 A.

Let w = weight of one cu. ft. water = 62.4 lbs.

w_1 = weight in air of 1 cu. ft. dry fill material.

ε = volume of solids in 1 cu. ft. of fill material.

$1-\varepsilon$ = volume of voids in 1 cu. ft. of fill material.

$\theta_2 = \tan^2(45° - \rho/2)$ the Rankine co-efficient in the formula for earth pressure, where ρ is the angle of repose of the wet material representing the maximum effectiveness of the back-fill.

The water pressure on a square foot, located h ft. below the water level, will be wh. This will be balanced by an equal and opposite hydrostatic pressure on the side of the fill, a condition which must prevail if the voids in the fill are sufficiently open to permit the water to circulate and wet the entire surface of sheet piling. In addition to the water pressure, there will be the effect of the back-fill, weighing $(w_1 - \varepsilon w)$ pounds per cu. ft. in water, and exerting a horizontal pressure $e = (w_1 - \varepsilon w)\theta_2 H$, in pounds per sq. ft., due to the submerged weight of the wet material.

The total unit pressure on the filled side, at a depth h below water level, thus becomes

$$p = wH + e = wH + \left[w_1 - \varepsilon w\right]\theta_2 H = w_1\theta_2 H + (1-\varepsilon\theta_2)wH \text{ ----- 53-A.}$$

This is the usual method of combining the hydrostatic pressure with the earth pressure and represents the combined effect due to full hydrostatic head increased by the pressure of the wet material acting under its submerged weight.

By assigning numerical values to the quantities in Eq. 53-A for the case of rip rap back fill of limestone, weighing 98 lbs. per cu. ft. and containing 35% voids, making $\varepsilon = 0.65$, then for an angle of repose $\varphi = 40^\circ$, the value $\theta_2 = \tan^2(45^\circ - \varphi/2) = 0.217$, giving

$$p = 98 \times 0.217 H + (1 - 0.65 \times 0.217) 62.4 H = 74.9 H \text{ lbs. per sq. ft.}$$

This is 20% greater than the water pressure in front of the wall. The triangular areas in Fig. 53-A, thus represent the total pressures on the two sides of the sheet piling. On the water side the total pressure is $62.4 \frac{H^2}{2}$ acting at $H/3$ above the bottom. On the side of the back fill the total pressure is $P = \frac{pH^2}{2} = 74.9 \frac{H^2}{2}$, also acting at $H/3$ above the bottom.

Case II Fine Material for Back Fill. The distinction which is here made between coarse and fine material for back fill, requires some explanation without which, the purpose of our classification would not be clear.

In speaking of coarse material the term was intended to apply only to rip-rap, crushed stone, gravel, or coarse sand, the voids of which are sufficiently open to allow water to circulate freely between the interstices and to run out or drain off whenever the material ceases to be submerged.

Fine material on the other hand may be described as having voids of such small dimensions that water cannot circulate between the particles and would be retained more, or less, even when the material is removed for a short time from the submerged state. Fine silty sand, clay, marl and mixtures of these with each other, or with coarser material, would be included under fine material and may retain from ten to thirty-five percent of water filling the voids.

The behavior of fine material, as here described, differs radically from that of the coarse variety, irrespective of the percentage of voids possessed by either class.

In the case of fine material a new physical law manifests itself in the property known as capillary attraction, implying a specific attraction between the water and the solid particles which are in such close proximity as to hold the water sufficiently to overcome the force of gravity. Such a material, once saturated, will lose little weight simply by drainage, and most of the retained water can be removed only by evaporation or pressure.

Therefore, when the back fill consists of fine material as here described, the above analysis of pressure given by Eq. 53-A, no longer applies and new assumptions become necessary. In no event can the full hydrostatic head now exert itself and we might deal with the problem as one of wet earth pressure which according to the Rankine formula might approximate hydrostatic pressure as a maximum. However, this may not be sufficiently safe for a completely saturated material such as we are now considering, and it would appear wise to estimate the maximum possible pressure value which might be encountered.

Such a maximum pressure value could be made up of the increment due to the dry weight acting with the highest Rankine coefficient for wet material, plus the hydrostatic pressure due only to the water filling the voids. For example take a fine silt having a dry weight $w_1 = 100$ lbs. per cu. ft., and $(1-\varepsilon) = 0.40$, with an angle of repose $\varsigma = 25^0$ for the same material when thoroughly wet. Then $\theta_2 = \tan^2(45^\circ - \frac{25}{2}) = 0.406$, and the combined pressure would thus become:

Pressure due to silt $= w_1\theta_2 H = 100 \times 0.406 H = 40.6 H.$

Pressure due to water in voids $= (1-\varepsilon) w H = 0.4 \times 62.4 H = \underline{25.0 H}.$

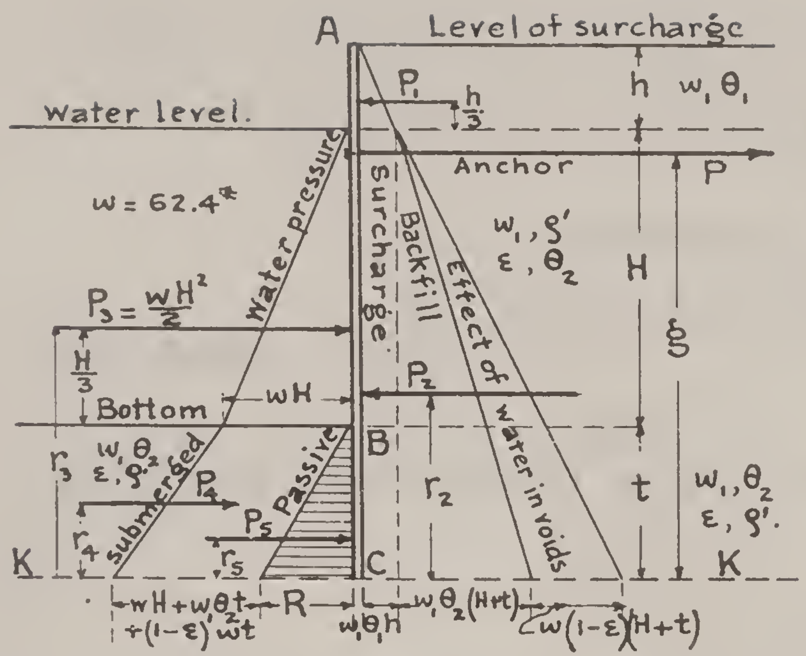
Total pressure of fill material $= p = 65.6 H.$

which is slightly more than full hydrostatic pressure of 62.4 H.

The total pressure thus found may be expressed as a formula thus $p = w_1\theta_2 H + (1-\varepsilon) w H$ ---------------------- 53-B.

The weight of this material when saturated, would be 100 + 25 = 125 lbs. p. cu. ft. and would exert a pressure of 125 x 0.406 H = 50.75 H pounds per sq. ft. according to Rankine's formula. This value is considerably under that due to water alone and is undoubtedly too low for saturated material. The pressure given by Eq. 53-B, would be 65.6 H, and this accords well with observations.

In accordance with the reasoning employed in arriving at Eq. 53-B, the solution of a sheet piling problem may be outlined as shown in Fig. 53-B. Quantities are per foot length of wall.



$P_1 = w_1\theta_1\frac{h^2}{2}$; $P_2 = \frac{H+t}{2}\left[2w_1\theta_1 h + \left(w_1\theta_2 + w(1-\varepsilon)\right)(H+t)\right]$;

$P_3 = w\frac{H^2}{2}$; $P_4 = wHt\left[w_1\theta_2 + w(1-\varepsilon)\right]\frac{t^2}{2}$; $P_5 = \frac{Rt}{2}$.

Fig. 53B.

w = wgt. per cu. ft. water = 62.4 lbs.
w_1 = dry weight of fill material
w_2 = wet weight of fill material
ε = volume of solids contained in 1 cu. ft. fill material, making volume of voids = $1-\varepsilon$ = 0.25 to 0.40 cu. ft.
$\theta_1 = \tan^2(45° - \varrho/2)$ for dry material.
$\theta_2 = \tan^2(45° - \varrho'/2)$ for wet material.
$\theta_p = \tan^2(45° + \varrho'/2)$ = coefficient for passive resistance for wet material.
P = pull on anchor rod.
P_1, P_2, P_3, P_4 are pressures in pounds for the pressure areas as indicated, and acting at the centers of gravity of the respective areas, with lever arms r above the base KK. Finally $R = w_2\theta_p t$ lbs. per sq. ft. passive resistance at the bottom of the sheet piling, making the triangular area $\frac{Rt}{2} = P_5$ represent the unbalanced force required to hold back the toe. The maximum value of $R = w_2\theta_p t$ should never be developed to more than one-half this value, so as to insure a factor of safety of at least two against failure by kicking out at the bottom.

In the above description of the problem everything may be regarded as known or assumed, except the anchor pull P and the bottom pressure R, and these may be regarded as the forces necessary to supply the conditions for equilibrium of the beam ABC. To find these two unknowns we may write two condition equations as follows:

$$\left.\begin{array}{l}\text{Sum of hor. forces} = 0, \text{ or } P - P_1 - P_2 + P_3 + P_4 + P_5 = 0 \\ \text{Sum of Mom. about } C = 0, \text{ or } Pg - P_1r_1 - P_2r_2 + P_3r_3 + P_4r_4 + P_5r_5 = 0\end{array}\right\} \text{-- 53-C.}$$

Noting that $P_5 = \frac{Rt}{2}$ and $P_5r_5 = \frac{Rt^3}{6}$ we obtain the following:

$$\left.\begin{aligned} P+\frac{Rt}{2} &= P_1+P_2-P_3-P_4 \\ P\underline{g}+\frac{Rt^2}{6} &= P_1r_1+P_2r_2-P_3r_3-P_4r_4 \end{aligned}\right\} \text{-------------- 53-D.}$$

which may be solved for R and P to obtain:

$$\left.\begin{aligned} R &= \frac{2}{t-\frac{t^2}{3\underline{g}}}\left[P_1+P_2-P_3-P_4-\frac{1}{\underline{g}}\left(P_1r_1+P_2r_2-P_3r_3-P_4r_4\right)\right] \\ \text{and} \quad P &= P_1+P_2-P_3-P_4-\frac{Rt}{2} \end{aligned}\right\} \text{---- 53-E.}$$

The method just outlined will now be illustrated by solving a complete problem.

EXAMPLE 53A. SHEET PILING REVETMENT WITH BACK ANCHOR.

The backfill is fine sand, assumed as dry above water level and saturated below water. The loads are per linear foot of piling. See Fig. 53A for dimensions.

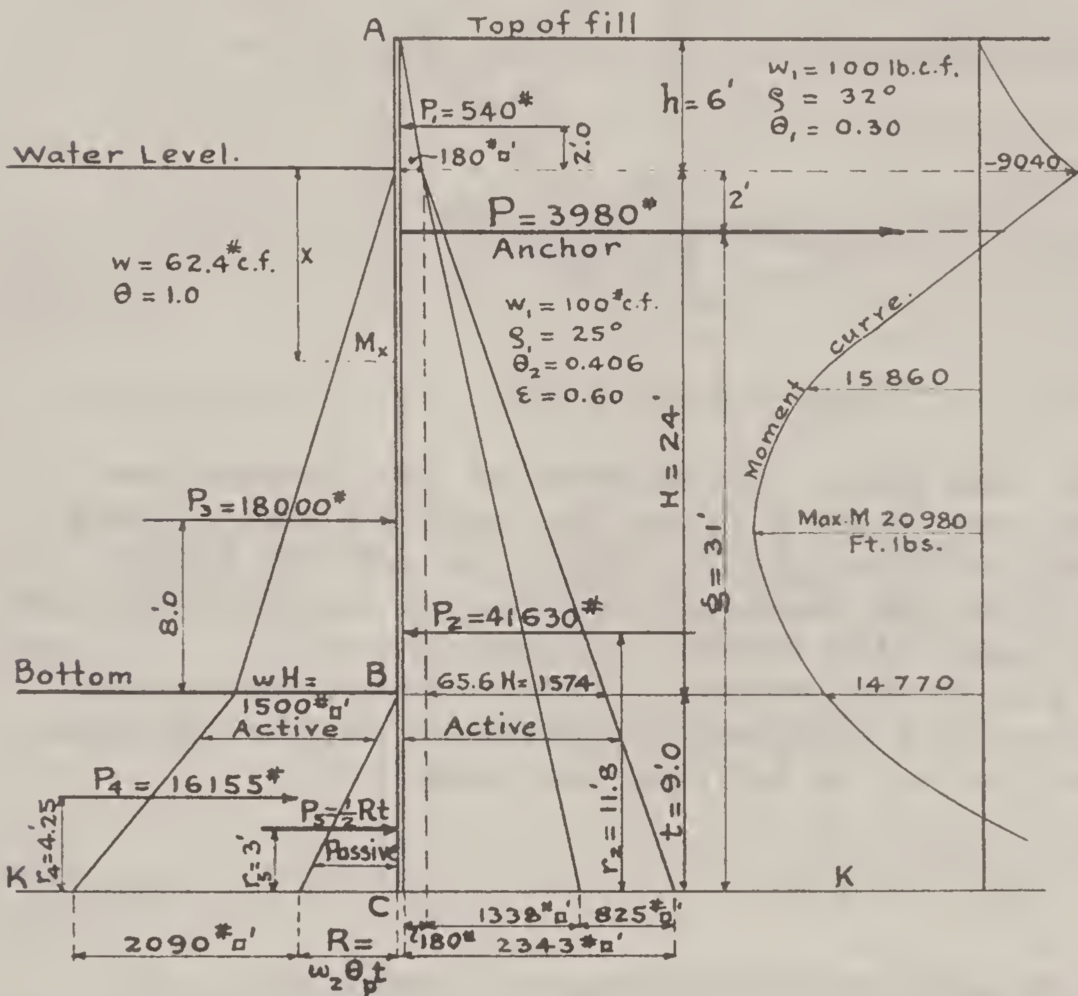


Fig. 53 C.

DATA.

$w = 62.4$ lbs. cu. ft. for water.

$w_1 = 100$ lbs. cu. ft. dry sand

$w_2 = 125$ lbs. cu. ft. sat. sand

$\varepsilon = 0.6$ solids per cu. ft.

$1 - \varepsilon = 0.4$ voids per cu. ft.

$\theta_1 = \tan^2(45° - \frac{32}{2}) = 0.30$

$\theta_2 = \tan^2(45° - \frac{25}{2}) = 0.406$

$\theta_p = \tan^2(45° + \frac{25}{2}) = 2.46$

$w_1\theta_1 h = 100 \times 0.30 \times 6 = 180$ lbs. sq. ft.

$w_1\theta_2(H+t) = 100 \times 0.406 \times 33 = 1338$

$62.4(1-\varepsilon)(H+t) = 25 \times 33 = 825$

$wH = 62.4 \times 24 = \quad 1500$

$w_1\theta_2 t + (1-\varepsilon)wt = \quad \underline{590}$

$wH + w_1\theta_2 t + (1-\varepsilon)wt = 2090^{\#}$ sq. ft.

$P_1 = 180 \times h/2 = \quad 540$ lbs

$P_2 = \frac{33}{2}(180 + 2343) = 41{,}630$ "

$P_3 = \frac{24}{2} \times 1500 = 18{,}000$ "

$P_4 = \frac{9}{2}(1500 + 2090) = 16{,}155$ "

$P_5 = \frac{1}{2}Rt = 4.5R$ lbs.

$P_1 + P_2 - P_3 - P_4 = 8{,}015$ lbs.

Eqs. 53E may now be solved for R and P by substituting the above data to obtain:

$$R = \frac{2}{9 - \frac{81}{3 \times 31}}\left[8{,}015 - \frac{1}{31}(540 \times 35 + 41{,}630 \times 11.8 - 18{,}000 \times 17 - 16{,}155 \times 4.25)\right] = 897 \text{ lbs. sq. ft.}$$

$$P = P_1 + P_2 - P_3 - P_4 - \frac{Rt}{2} = 8{,}015 - 4.5 \times 897 = 3980 \text{ lbs.}$$

This gives for $P_5 = 4.5R = 4{,}036$ lbs., as against an available value of $w_2\theta_p\frac{t^2}{2} = 125 \times 2.46 \times \frac{9^2}{2} = 12{,}454$ lbs., representing a factor of safety of $\frac{12454}{4036} = 3.1$.

The bending moment on the sheetpiling will now be found by writing the moment for any point x feet below water level. Thus:

$M_x = \frac{wx^3}{6} + P(x-2) - P_1(x+2) - \frac{65.6\,x^3}{6} - \frac{180\,x^2}{2}$, and substituting the above values of P and P_1, get $M_x = 10.4x^3 + 3980x - 7960 - 540x - 1080 - 10.9x^3 - 90x^2$, which simplifies into

$$M_x = -0.5x^3 - 90x^2 + 3440x - 9040 \qquad \text{53A.}$$

To find the value of x for which M_x is maximum, equate the first differential derivative of Eq. 53A to zero and solve for x. This gives:

$\frac{\partial M_x}{\partial x} = -1.5x^2 - 180x + 3440 = 0$, from which $x = +16.7$ ft., and with this value of x, Eq. 53A gives maximum $M_x = 20{,}980$ ft. lbs. The moments for $x = 0$, $x = 2$, $x = 10$ and $x = 24$ ft. were also found from Eq. 53A, and are plotted in Fig. 53C to show the entire moment curve.

Art. 54. Resistance developed by back anchors. Let the plate BC represent a continuous anchor, submerged below the ground as shown in Fig. 54-A. All forces are per unit length of anchor plate.

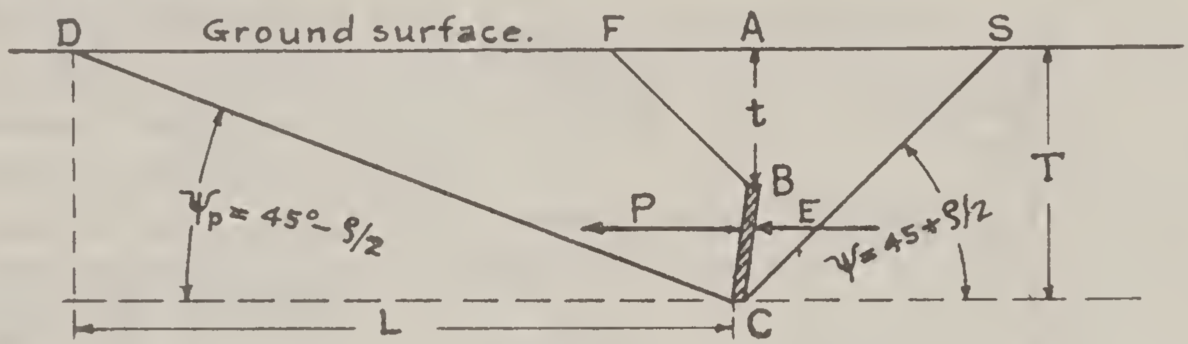


Fig. 54 A.

Let P represent the safe resistance of the anchor per foot of length which may result from the passive resistance of the earth in front of the plate, less the active earth pressure $E = \frac{w}{2}\theta(T^2 - t^2)$ on the back of the plate, neglecting the active pressures on each side of the plane AB = t.

Introducing a factor of safety n into the passive resistance $\frac{w}{2}\theta_p(T^2 - t^2)$, the following equation may be written for a length of one foot of a continuous anchor,

$$P \lesseqgtr \frac{w}{2}\left(\frac{\theta_p}{n} - \theta\right)(T^2 - t^2) \quad \text{---------- 54-A.}$$

where P is assumed to act at the middle of plate BC.

When the force P approaches the condition for failure then the prism DFBC must be moved forward and upward on the plane of rupture DC, while the prism FSCB would slide on the planes of rupture CS and FB.

For an anchor plate of length b, the passive resistance becomes relatively greater, owing to the lateral spread of the passive earth resistance at the two ends.

The safe pull on an anchor plate of area b(T-t) is

$$P \lesseqgtr \frac{wb}{2n}\theta_p T^2 - \frac{wb\theta}{2}\left[T^2 - \left(\frac{n-1}{n}\right)t^2\right] + \frac{w\theta}{3n} L T^2 \tan\varrho \quad \text{---- 54-B}$$

When the clear distance between the edges of such anchor plates has the value d, or less, as given by

$$d \lesseqgtr \frac{2w\theta L T^2 \tan\varrho}{3\left[\theta_p T^2 - \theta t^2\right]} \quad \text{---------- 54-C.}$$

then the effectiveness of the isolated anchors becomes equal to that obtainable by a continuous plate as per Eq. 54-A.

Example 54-A. Supposing the anchor plate of height 6 ft. is submerged 5 ft. below the surface in ordinary earth with $w = 100$ lbs. per cu. ft. and $\varphi = 30^0$, what is the safe resistance per ft. length of anchor? This makes $t = 5$ ft., $T = 11$ ft., $\theta = \tan^2(45-\varphi/2) = 0.333$ and $\theta_p = \tan^2(45+\varphi/2) = 3.00$. Assuming a factor of safety $n = 2$, then Eq. 54-A gives :

$$P = \frac{100}{2}\left(\frac{3.00}{2} - 0.333\right)(11^2 - 5^2) = 5620 \text{ lbs. per lin. ft.}$$

Now suppose we have isolated anchor plates of length $b = 10$ ft. find the safe pull for one such plate. With the same data as above and $n = 2$, then from Eq. 54-B for $L = T\cot(45° - \varphi/2) = 19.05$ ft., we have:

$$P = \frac{100\times 10}{2\times 2}\times 3.0\times 121 - \frac{100\times 10\times 0.333}{2}\left[121 - \frac{1}{2}\times 25\right] + \frac{100\times 0.333}{3\times 2}\times 19.05\times 121\times 0.577$$

or 80,060 lbs. which represents say 8006 lbs. per foot of length as against 5620 lbs. previously found for a continuous plate.

The distance apart d, which such plates can be spaced without loss of effectiveness, is given by Eq. 54-C as

$$d \leqq \frac{2\times 100\times 0.333\times 19.05\times 121\times 0.577}{3\left[3.0\times 121 - 0.333\times 25\right]} = 13.9 \text{ ft.}$$

While these values are necessarily quite approximate the above formulas undoubtedly lead to far more rational designs than are commonly made.

Art. 55 Length of Anchor Rods and the location of Anchor plates will now be considered, in order that the anchorage may function properly in supporting a wall subjected to earth pressure.

There may be three possible cases which would govern the effectiveness of anchor plates as follows:

1. Anchors located between the wall and the plane of rupture.
2. Anchors located between the plane of rupture and the natural slope.
3. Anchors located back of the natural slope.

Anchors located in the pressure wedge between the wall and the plane of rupture cannot have any practical value and deserve no further consideration.

Anchors located at C′, beteeen the surface of rupture BD and the natural slope BF, cannot in themselves offer any support to a wall AB, since the back-fill may slide on a surface which

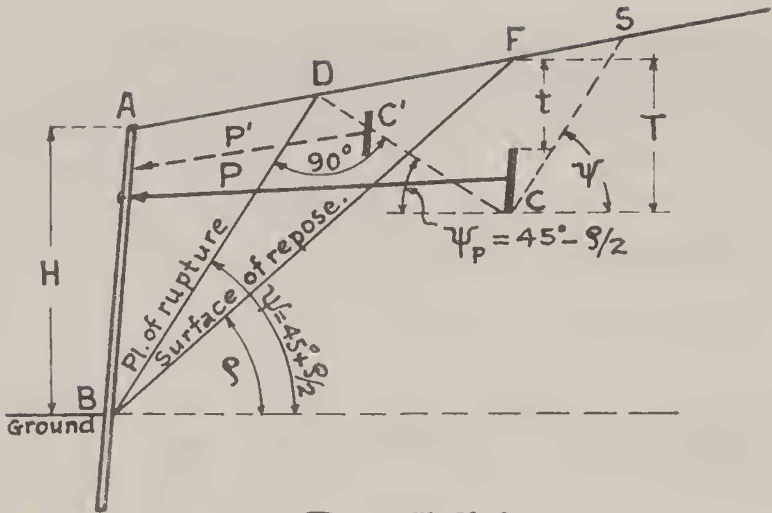


Fig. 55A.

is steeper than the natural slope without developing any appreciable passive resistance. See Fig. 55-A.

However, when the wall is otherwise supported as by driving the sheet piling into the ground at B, or in case of a masonry wall adequately supported on a good foundation, there may be a divided responsibility between such supports and the back-anchor even when each alone would be inadequate to furnish the necessary stability. It is therefore, not advisable to place much dependence on anchors of this class.

The only safe anchor is one which is located behind the natural slope as shown at C Fig. 55-A, and the distance back from the wall should be such that the intersection of the two planes of rupture BD and DC as at D, falls practically on the surface AS, or only slightly below.

The formulas previously given for safe pull on the anchor plate are then applicable.

For level earth surface and vertical sheet piling the length of anchor should be not less than

$$l \geqq H \tan(45° - \rho/2) + T \tan(45° + \rho/2) \text{ ---------------- } 55\text{-A.}$$

CHAP. 11. WAVE PRESSURE, SEA WALLS, BREAKWATERS.†

Art. 56 On the Nature of the Problem. Sea walls and Breakwaters are exposed to the action of waves generated by high winds, sweeping over a considerable expanse of open water. The size of a wave for any particular locality depends on the velocity of the wind, duration of the storm, depth of water, and the greatest distance over which the wind can act, provided the water is of sufficient depth for wave formation. The distance from the windward shore is called the fetch.

Assuming a constant direction and velocity of wind, the height of a wave will increase from zero at the windward shore to some limiting height which the assumed wind velocity will just maintain. When the fetch is sufficiently great and the depth of water is ample, the wave will attain the limit of height before reaching the leeward shore, continuing with a constant height until some resistance is encountered in shallower water or against a structure of some sort.

As a wave increases in size according to the distance out from the windward shore, its length also increases and so does its velocity of propagation. When the balance is finally attained between the wind velocity and the wave dimensions, then the velocity of the wave no longer increases, but may continue without reduction for long distances even after the wind has subsided.

Assuming for the present that the height of a wave may be estimated for a given fetch and wind velocity, as established for a given location where it is proposed to construct a sea wall or breakwater, then the next step is to evaluate the probable mass and velocity of such a deep water wave and finally trace its retardation and diminished dimensions as it advances into shallower water previous to striking the wall where its moving mass is stopped.

In case the wall is so situated that the maximum wave cannot strike normally, then there will be a further reduction in resolving the wave force normally to the face of the wall.

Having finally decided on the probable size of wave which may be expected to arrive in front of a wall, we next estimate the height to which this wave will be piled up when completely obstructed, and then decide on the height to which the wall is to be built. If the wall is not carried up to the full height of the completely obstructed wave, then a portion of the wave will go over the top of the wall and the wave will be only partially obstructed.

When a wave is partially or completely stopped by a wall, then a portion, or all, of its energy is converted into a dynamic force, which in turn represents the static equivalent

† This Chapter was published in Transactions, Am. Soc. C.E., 1936.

of the expended energy. The wall structure must be capable of resisting this static equivalent without exceeding certain requirements of structural stability and safety.

The various phases of the problem as above outlined will now be examined in detail, to show how each element may be quantitatively evaluated in a logical sequence. It should be remembered however, that all formulas employed in this connection are based on observations which are in themselves quite approximate, while a theoretical basis is only remotely possible.

Most of the data employed herein are taken from Professional Papers No. 31 of the Corps of Engineers, U.S. Army, on "Wave Action in relation to Engineering Structures", by D.D. Gaillard, Capt. Corps of Engrs., 1904. The application of Capt. Gaillard's observations, supplimented by others made in 1915 at Toronto, Ont., was developed by the author in connection with the Toronto Harbor work.

Art. 57 Height of Waves in Terms of Wind Velocity and Fetch. For a given wind velocity in miles per hour, and a fetch D in statute miles, the height, h in feet, of a wave, may be estimated from the formulas

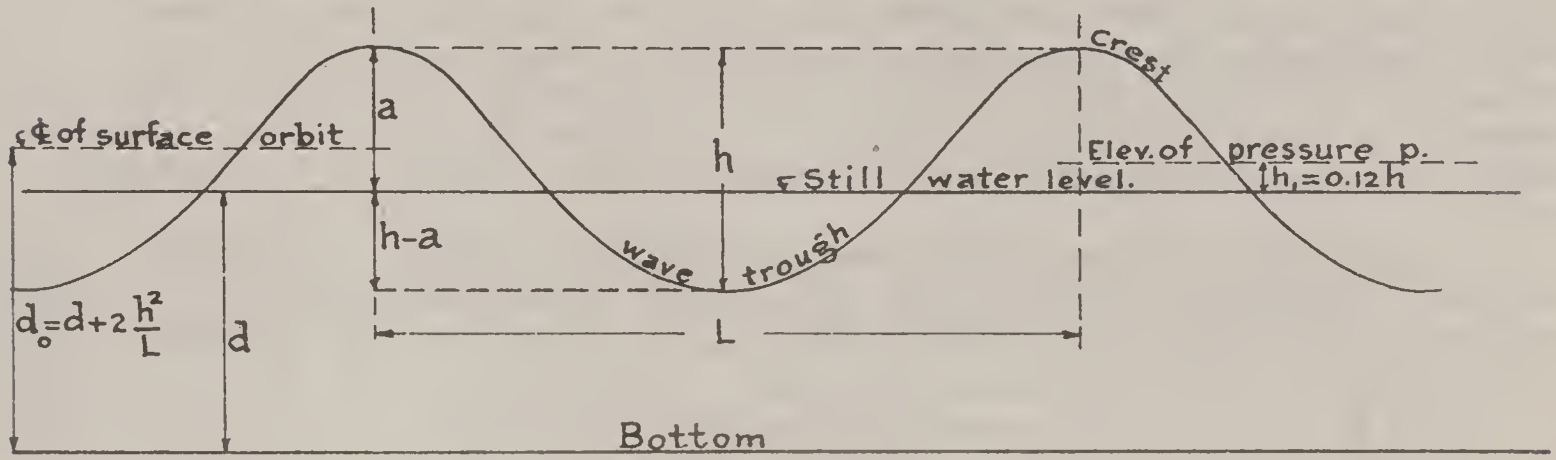
$$\left.\begin{array}{ll} h=0.17\sqrt{VD} & \text{for values of D greater than 20 miles} \\ h=0.17\sqrt{VD}+2.5-\sqrt[4]{D} & \text{for D less than 20 miles} \end{array}\right\} \text{--57-A}$$

which are based on the formulas of Thomas Stevenson, after introducing wind velocity as a variable, and employing statute miles instead of nautical miles.

These formulas apply only to inland lakes where the fetch is not likely to exceed the distance subjected to a violent wind in a single direction.

With few exceptions ocean waves rarely exceed 45 feet, corresponding to a wind velocity of 75 miles per hour and a fetch of about 935 miles, which represents an unusual combination of circumstances. Ocean storms are generally more or less local, and do not cover more than 50 to 100 miles. Since the oceans are of much greater extent than any possible storm, the fetch is restricted to the storm area which may be anything from a few miles to say five hundred miles, and no formula can be made to apply to such indeterminate conditions. Hence, the only reliable data relative to the height of ocean waves must be collected by direct observations for any given locality. An extensive collection of ocean wave observations is given on p. 76 of Capt. Gaillard's treatise.

Art. 58 Length of Wave in terms of height and wind velocity. The maximum wave has the minimum length ratio and as the wind subsides the height diminishes and the length ratio increases. Fig. 58-A gives certain dimensions for waves which will be used throughout the present chapter.



t = the period of the wave, or time in seconds for the crest to travel L ft.

$v = L/t$ = the velocity of wave propagation in feet per sec.

Fig. 58-A.

a. The ratio L/h varies between rather wide limits depending on the velocity of the wind V, the duration of a storm, and the depth of water in which the wave is formed. According to observation by Capt. Gaillard in fresh and relatively shallow water, this ratio varies between the limits 9.1 and 15. A formula here proposed, expresses this ratio in terms of wind velocity V as follows:

$$\frac{L}{h} = \frac{840}{V} \quad \text{or} \quad L = 840\frac{h}{V} \qquad \text{58-A}$$

This gives the following values for various wind velocities.

V=	75	70	65	60	55	50	45	40	35	30.
$\frac{L}{h}$=	11.2	12.0	12.93	14.0	15.3	16.8	18.7	21.0	24.7	28.0

For ocean waves the following are given by Dr. G. Schott

Moderate wind,	Beaufort Scale	5	= 28 m.p.hr	L = 33 h.
Strong wind,	" "	6 to 7	= 35 " " "	L = 20 h.
Storm	" "	9	= 56 " " "	L = 17 h.

The ocean waves are thus relatively longer than waves on the Great Lakes, as might be expected.

b. Height of wave above still water level. Since the crest of a wave is above the still water level and the trough is below that level it is sometimes erroneously assumed that the still water level represents the mean of the two. However, many observations made by various observers show the crest to be about 2h/3 above the still water level in deep water, and raising still higher in shallow water, a fact which must not be disregarded when examining the safety of a structure.

According to many observations made by Capt. Gaillard on Lake Superior, the height of a wave crest above still water level before the wave breaks, is given by the formula:

$$\left.\begin{aligned} a &= \frac{h}{2} + \frac{h^2}{L} \text{ for deep water waves, with } d > 1.84h \\ a &= \frac{h}{2} + 2\frac{h^2}{L} \text{ for shallow water waves, with } d < 1.84h \end{aligned}\right\} \text{--- 58-B.}$$

For deep water ocean waves, Rankine gives the formula

$$a = \frac{h}{2} + 0.785\frac{h^2}{L} \text{ ---------------------- 58-C.}$$

The term still water level as used here, represents a line for which the sectional area of the wave ridge is equal to that of the wave hollow.

c. Depth of water in which a wave breaks. While deep water waves may break due to the action of increasing wind and other causes, yet this is invariably the case when a wave reaches water of insufficient depth.

A knowledge of this minimum depth is necessary in determining the maximum wave which may be expected to arrive at a certain structure located in shallow water.

Undoubtedly the roughness of the bottom has much to do with this subject and it would seem that the depth in which a wave breaks is much less for a rough bottom than for a smooth sandy bottom.

For 55 observations at the Duluth Canal by Capt. Gaillard, for waves from 7 to 13 ft. in height and a sandy bottom sloping about 1:40, the waves broke in a depth equal to 1.72 h. Similar observations near Presque Isle Pt. and near Grand Marais, Lake Superior, with rough bottom and waves from 6 to 9 ft. high, the waves broke in a depth equal to 1.3 h.

For ocean waves at St. Augustine, with strong wind blowing in the direction of wave travel, the waves broke in a depth of 1.25 h.

Hence, calling d_1 the depth in which a wave breaks, and h the height of a deep water wave from trough to crest, the following average relations may be accepted for storm waves with wind in the direction of the wave travel.

$$\left.\begin{array}{llll} \text{Lake Superior,} & \text{sandy bottom,} & \text{wind } V = 40 \text{ m.,} & d_1 = 1.84h \\ \text{"} & \text{"} & \text{" } V = 26 \text{ m.,} & d_1 = 1.42h \\ \text{"} & \text{"} & \text{gentle breeze,} & d_1 = 1.34h \\ \text{"} & \text{rough bottom,} & \text{wind } V = 30 \text{ m.,} & d_1 = 1.30h \\ \text{Ocean} & \text{sandy bottom,} & \text{wind } V = 40 \text{ m.,} & d_1 = 1.25h \end{array}\right\} \text{58-D.}$$

Art. 59. Dynamic Properties of Waves. The previous article dealt with wave dimensions, and the relations existing between these and the actuating force. The present article will be devoted to the dynamic properties including the velocity of propagation, energy of a wave, wave pressure, and height of a completely obstructed wave.

a. Velocity of propagation. According to Rankine for long waves or waves in water that is very shallow compared with the wave length, the velocity is given by the formula:

$$v = \sqrt{g\left(d + \tfrac{3}{4}h\right)} = 5.68\sqrt{d + \tfrac{3}{4}h} \quad \text{--------------- 59-A.}$$

This formula according to Capt. Gaillard, invariably gives results in excess of observed velocities.

The theoretical velocity for deep water waves with $d > L/2$ is:

$$v = \frac{L}{t} = \sqrt{\frac{gL}{2\pi}} = 2.26\sqrt{L} \quad \text{------------------ 59-B.}$$

For shallow water waves with $d < L/2$, the formula becomes

$$v = 2.26\, c\sqrt{L} \quad \text{when } \frac{d_0}{L} = \frac{d}{L} + 2\left(\frac{h}{L}\right)^2 \text{---------- 59-C.}$$

where c has the following values depending on d_0/L

d_0/L =	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40
c =	0.552	0.746	0.858	0.922	0.958	0.977	0.988	0.994

The wave length L and velocity v are materially altered when the depth gradually decreases toward a shoal or shore, and Capt. Gaillard proposed an empiric formula based on a number of observations made at Duluth Canal, where the waves entered the canal at 26 ft. depth, while the portion of wave

outside the canal entered shallow water, gradually shoaling to 3.3 ft. The formula is:

$$v_2 = 0.9\, v \sqrt[4]{\frac{d_2}{d}} \quad \text{with} \quad h_2 = h\sqrt{\frac{d_2}{d}} \text{ ----------------- } 59\text{-D.}$$

in which v is the velocity of the wave approaching from a depth d which should not be greater than $L/2$, and v_2 is the reduced velocity when the depth has shoaled to the lesser depth d_2. A wave thus retarded may have a reduced height of $h_2 = h\sqrt{\frac{d_2}{d}}$ as per Eq. 59-D.

In applying the formulas 59-D it is well to choose a deep water depth about $L/5$ instead of $L/2$ because the dimensions of a deep water wave are not appreciably altered until after the shoaling has become effective.

b. The energy of a wave. The theoretical energy for a deep water wave of length L feet, height h feet, and breadth of one foot, for fresh water, is given by

$$E = 7.8\, L h^2 \left[1 - 4.935\left(\frac{h}{L}\right)^2\right] \text{ft. lbs.} \text{ ---------------- } 59\text{-E.}$$

For salt water the coefficient 7.8 becomes 8.0.

For shallow water waves the energy decreases from 2 to 10% below the value given by Eq. 59-E and the appropriate formula becomes rather complicated. No particular use is made of this formula so the matter will not be investigated further.

c. Wave Pressure. Having enumerated the various steps in arriving at the probable dimensions of storm waves for a given locality, the important question confronting the designing engineer, is to appraise with some reasonable degree of accuracy, the force which such a wave develops in striking a proposed structure.

The most satisfactory way of attacking this problem is by direct measurement of the force expended by impinging waves on a solid structure carrying a series of selfregistering dynamometers.

Dynamometer measurements were carried on for many years by Thomas Stevenson at Skerryvore Rocks in the Atlantic Ocean in 1843-4, also at Dunbar Harbor in 1858.

Several attempts to measure wave pressures on the Great Lakes were made at various times, but without gaining much information of value. The experiments made, in 1901-3, by Capt. Gaillard on Lake Superior, stand as the most valuable contribution to this subject.

Before mentioning any experimental results, the following conclusions arrived at by Capt. Gaillard as a result of his careful and painstaking work are here quoted.

1. That the impact of a wave does not at all resemble that of a solid body.

2. That the pressures indicated by the types of dynamometers heretofore used are due to dynamic action only.

3. That these pressures apparently conform to the hydrodynamic laws governing the action of a current flowing normally against a submerged plane.

4. That a mass of water in air projected with a certain velocity against a plane surface can produce no greater pressure than would be caused by the steady flow against this surface of a jet of equal cross-section having the same velocity and striking at the same angle.

From the above statements, it apparently follows by inference.

5. That a mass of water projected against a submerged plane surface of considerably smaller area than the cross-section of the mass, can produce no greater pressure than would be caused by the steady normal flow at the same velocity of a current against a submerged plane surface of equal area and similar to the first.

6. That as the most destructive waves act for an appreciable period, the pressures which they exert can properly be measured by suitably constructed dynamometers.

Observations made with spring dynamometers on the outer end of the South pier, Duluth Canal, Lake Superior, by Capt. Gaillard, are given as representing the best typical wave measurements available. See Table 60-A.

The formula for pressure on a submerged plate according to Dubuat is

$$p = k w \frac{v^2}{2g} \text{ lbs. per sq. ft.}$$

in which v is the velocity of forward motion of the plate.

The total velocity of the striking wave consists of the combined velocity of propagation v and the maximum orbital velocity v_0 of a wave particle, so that the above formula should be modified to represent the maximum value

$$p = k w \frac{(v + v_0)^2}{2g} \text{ lbs. per sq. ft.} \quad \text{------------------ 59-F.}$$

where k is an empiric coefficient evaluated from Capt. Gaillard's observations for the Great Lakes as 1.30 to 1.71 for winds from 30 to 70 miles per hour. For ocean storm waves k may be taken as 1.8. For sweet water w= 62.4 lbs. cu. ft. and for

salt water w = 64.4 lbs. while the acceleration due to gravity is g = 32.2 sec. ft.

According to Eq. 59-C the velocity of wave propagation for waves formed in any depth of water is :

$$v = 2.26\, c \sqrt{L}\ , \text{ where } c = \sqrt{\frac{b_s}{a_s}} \quad \text{with } \frac{b_s}{a_s} = \text{the ratio of the}$$

semi-axes of the surface orbits as previously given.

The maximum orbital velocity of a wave particle is

$v_o = m\, c \sqrt{\frac{2\pi g}{L}}$, where $m = \frac{\beta h}{2}$ = the semi-major axis of the elliptical orbit and β is the ratio of the axes of the elliptical orbits.

The numerical values of c and β, given in Table 59-A, render these expressions applicable to the solution of problems.

Table 59-A - Values of c and β for values of d_o/L .

$\frac{d_o}{L} = \frac{d}{L} + 2\left(\frac{h}{L}\right)^2$	c	β	$\mu = c\beta$	
0.05	0.552	3.286	1.814	
0.10	0.746	1.796	1.340	
0.15	0.858	1.358	1.165	
0.20	0.922	1.177	1.085	
0.25	0.958	1.090	1.044	
0.30	0.977	1.047	1.023	
0.35	0.988	1.025	1.013	
0.40	0.994	1.013	1.007	
0.45	0.997	1.007	1.004	
0.50	0.998	1.004	1.002	

Introducing the numerical values of π and g, the above velocity formulas become:

$$v = 2.26\, c \sqrt{L} \quad \text{and} \quad v_o = mc\sqrt{\frac{2\pi g}{L}} = \frac{c\beta h \times 14.22}{2\sqrt{L}} = 7.11\, \mu \frac{h}{\sqrt{L}}\ ,$$

and with values of k, w and g substituted into Eq. 59-F, the maximum unit wave pressure becomes:

$$\left.\begin{aligned} &p = k\frac{w}{2g}\left(v + v_o\right)^2 \text{ lbs. per sq. ft.} \text{ ----------} \\ &p = 1.71\left(v + v_o\right)^2 \text{ for fresh water storm waves..} \\ &p = 1.80\left(v + v_o\right)^2 \text{ for salt water storm waves...} \\ &\text{where } v + v_o = 2.26\, c\sqrt{L} + 7.11\, \mu\frac{h}{\sqrt{L}} \text{ ---------} \end{aligned}\right\} \quad \text{59-G.}$$

with values of c and μ as given in Table 59-A. The values given for k are maximum for 75 mile wind and may become as low as 1.3 for 30 mile wind.

When $d_o/L \geqq 0.5$ then $c = \mu = 1.00$, which is the condition for deep water waves.

Having determined the maximum pressure exerted by a striking wave, it is necessary to know the elevation above still water level at which this maximum occurs, also the height to which a completely obstructed wave acts.

The maximum wave pressure occurs at a height h_1 above still water level, the theoretical value of which is:

$$h_1 = 0.785\frac{h^2}{L}, \text{ while } {}^{\bullet}h_1 = 0.12h \text{ gives better values} \text{-- 59-H.}$$

When a wave is completely obstructed by a vertical surface of sufficient height, the wave crest is raised to a height equal to $h_o = 2a$ above the still water level, and at this height the pressure becomes zero.

With this data it is now possible to design the entire wave pressure area and thus evaluate the total force to be resisted by a proposed structure.

d. Oblique waves striking a breakwater produce less pressure than when striking normally. Using the relations indicated in Fig. 59-A, the wave pressure p, Eq. 59-F, is reduced to a value

$$p_n = p \sin^2\alpha \text{ ---------------------------- 59-H.}$$

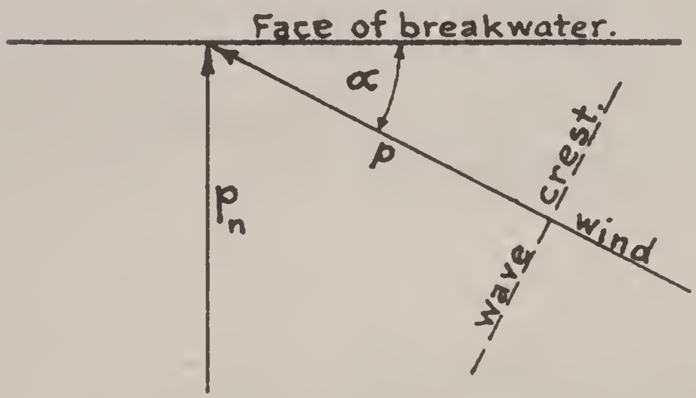


Fig. 59-A.

Observations made by the Author at Toronto, Ont., in 1915-16 gave the following result:

$p = 1460$ lbs. per sq. ft.

$p_n = 820$ lbs. per sq. ft.

$\alpha = 50°$, $\sin^2\alpha = 0.587$

By formula $p_n = 1460 \times 0.587 = 857$ lbs.

Art. 60. Wave pressure observations Lake Superior made in 1901-2 by Capt. D. D. Gaillard, Table **60-A**, are given as representing the best available data and will serve as a pattern of wave pressure curves. It is regrettable that no accurate wind velocities were recorded although it was stated that moderate wind, from 20 to 35 miles, produced these waves at Duluth. The pier carrying the dynamometers was not of sufficient height to completely obstruct the larger waves, so that for waves over 13 ft. high the recorded pressures are for partially obstructed waves.

Table 60-A Wave Pressure Measurements, Duluth Canal, L.Superior.

DATE	Elev. still water level	Observed Max. wave dimensions Depth d = 25 ft.			Max. Dynamometer Readings. Pounds per Sq. ft.			Computed		
		Height	Length	Velocity	Elevations of Dynamometers			$\frac{d_o}{L}$ ×	$v_o = 7.11\mu \frac{h}{\sqrt{L}}$	$p = 1.3(v+v_o)^2$
		h	L	v	+0.07 ft	+3.74 ft.	+7.01 ft.			
	ft.	ft.	ft.	sec.ft.					sec.ft.	lbs.sq.ft.
1901 July 24	+ 1.7	12	150	24.2	250	1150	1030	0.180	7.8	1330
Aug. 9	+ 1.9	12	130	24.2	370	1190	780	0.210	8.0	1348
Oct. 9	+ 1.9	13	150	29.6	0	1615	1260	0.180	8.4	1876
Nov. 22	+ 1.4	14	150	27.2	0	1605	1605	0.184	9.0	1705
Sep. 24	+ 1.9	16	250	33.2	1630	2255	2050	0.110	9.3	2344
1902					+7.04 ft.	+12.57 ft.	+16.18 ft.			
Oct. 25	+ 1.7	16	200	30.0	1755	1335	0	0.138	9.5	2026
Dec. 20	+ 1.7	16	210	31.0	1700	1430	515	0.131	9.4	2120
Nov. 12	+ 1.7	18	250	32.0	2370	2195	1370	0.110	10.5	2344

× $\frac{d_o}{L} = \frac{d}{L} + 2\left(\frac{h}{L}\right)^2$. Values of μ are taken from Table 59A.

The coefficient 1.3, in the formula $p = 1.3(v+v_o)^2$, was chosen because none of the above waves were caused by extreme storms, while the possible maximum coefficient 1.71 given for Eq. 59-F is more probable for winds from 70 to 75 miles per hour with shorter waves.

In order to arrive at the total wave pressure to be resisted by a sea wall or breakwater, it becomes necessary to design the complete wave pressure curve from data previously calculated for a given wind velocity and fetch.

We thus obtain the wave dimensions h, L, a and finally determine v, v_o and p. It still remains to find the total wave pressure for a completely or partially obstructed wave of these dimensions, and this must be approximated by patterning after some typical set of observations like those shown on Fig. 60-A.

By selecting the maximum pressure values for a 16 ft. wave from Table 60-A, into a composite pressure curve, the section shown in Fig. 60-A was obtained. The crest of this wave was 10.8 ft. above the still water level, and this crest was raised to a height of 18.3 ft. for the partially obstructed wave with the trough at 5.18 ft. below the still water level. Had the wave been completely obstructed, its crest would have reached a height $h_o = 2a = 21.6$ ft. The straight line AC is drawn to represent the maximum pressure of the completely obstructed wave with the maximum pressure at $h_1 = 0.12h = 1.9$ ft. above the still water level. The pressure curve is completed by drawing the curved line CB. The total wave pressure for the completely obstructed wave is represented by the triangular area ABC $= P_m = 35{,}390$ lbs., acting at the center of gravity of the pressure area which is 5.8 ft. above still water level.

For the partially obstructed wave as observed, the total wave pressure figures 33,340 lbs., acting at 5.42 ft. above the still water level.

A composite wave pressure curve as above described, represents the plotting of the maximum pressures recorded, for the duration of a storm, on the several dynamometers located at different elevations. It is evident that this record is not produced by any single wave, but that waves of various sizes contribute to produce the maximum pressures recorded. Hence, the total pressure area of such a curve is undoubtedly greater than the actual and thus includes a certain factor of safety.

A COMPOSITE WAVE PRESSURE CURVE BASED ON OBSERVATIONS FOR A 16 FT. WAVE IN LAKE SUPERIOR AT DULUTH.

Observations made in 1901-2 by Capt. D. D. Gaillard, Corps of Engrs., U.S.A.

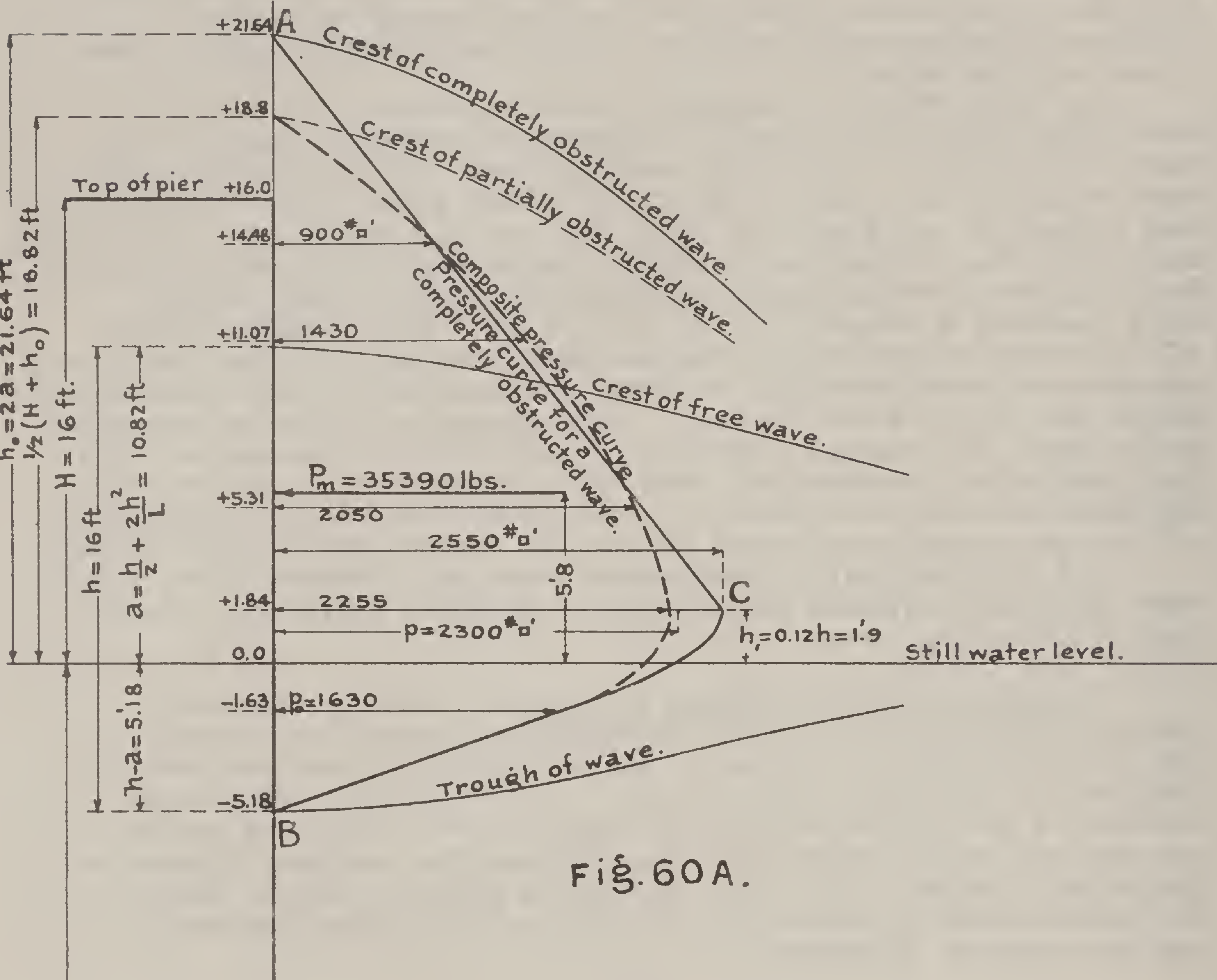


Fig. 60A.

DATA.

Fetch 260 m. N.E. wind velocity 35 m. per hr. $h = 0.17\sqrt{35 \times 260} = 16.5$ ft.

Observed $h = 16$ ft. $L = 210$ ft. $v = 31$ sec. ft. $d = 25$ ft.

$a = \frac{h}{2} + \frac{2h^2}{L} = 10.82$ ft., $h - a = 5.18$ ft., $h_o = 2a = 21.64$ ft.

$d_o/L = 0.13$, $C = 0.81$, $\mu = 1.22$, $v_o = 7.11 \times 1.22 \times \frac{16}{\sqrt{210}} = 9.57$ sec. ft.

$p = 1.4\,(31 + 9.57)^2 = 2300$ lbs. sq. ft. at $h_1 = 0.12h = 1.9$ ft. above still water.

$P = 33{,}340$ lbs. for composite pressure curve.

$P_m = 35{,}390$ lbs. for completely obstructed wave.

$p_o = 1600$ lbs. sq. ft. located halfway between B and C, may be taken at $0.72\,p$ when designing a wave pressure curve.

Wave pressure observations, Lake Ontario, made by the author for the Toronto Harbor Commissioners at Toronto, during a storm on Nov. 19, 1915. The storm commenced at 5 P.M. on Nov. 18, with east wind, and reached a maximum of 42 miles per hour about 7 A.M. on Nov. 19.

The observations were made at 10 A.M. with the wind at 30 miles after which the wind veered to the S.E. and died at 2 P.M. after swinging to the south.

The pressures were recorded on spring dynamometers mounted on a solid crib situated about 700 ft. off shore in about 8 ft. of water. At a distance of about 1200 ft. from the crib the water deepened rapidly to about 40 ft. In front of the crib, three piles were driven at about 150 ft. centers forming an equilateral triangle and carrying gage boards for observing wave height, also direction and velocity of wave crests. The results of these observations are shown on Figs. 60-B and C.

The deep water wave dimensions Fig. 60-D were estimated for a 30 mile wind and a fetch of 100 miles, representing something like an average condition prior to reaching the maximum. The still water level was at El. 245.0 out in deep water and began to raise when the depth was reduced to 17 ft. where the waves broke, and finally reached the Elev. 247.2 on the beach.

The reduced wave after entering shallow water, and producing the record shown by Fig. 60-C, had a height of 5.5 ft., length 118 ft., and velocity 19.8 ft. per sec. These dimensions and pressures were checked by applying the formulas previously given to show the agreement between computed and observed values.

It is thus seen that for a given condition of wind, fetch and depth in which a proposed break-water is to be built, the wave pressure curve may be estimated with sufficient accuracy to test the stability of the proposed structure.

WAVE OBSERVATIONS ON LAKE ONTARIO AT TORONTO,

STORM NOV. 19, 1915, USING SPRING DYNAMOMETERS.

By David Molitor, C.E.

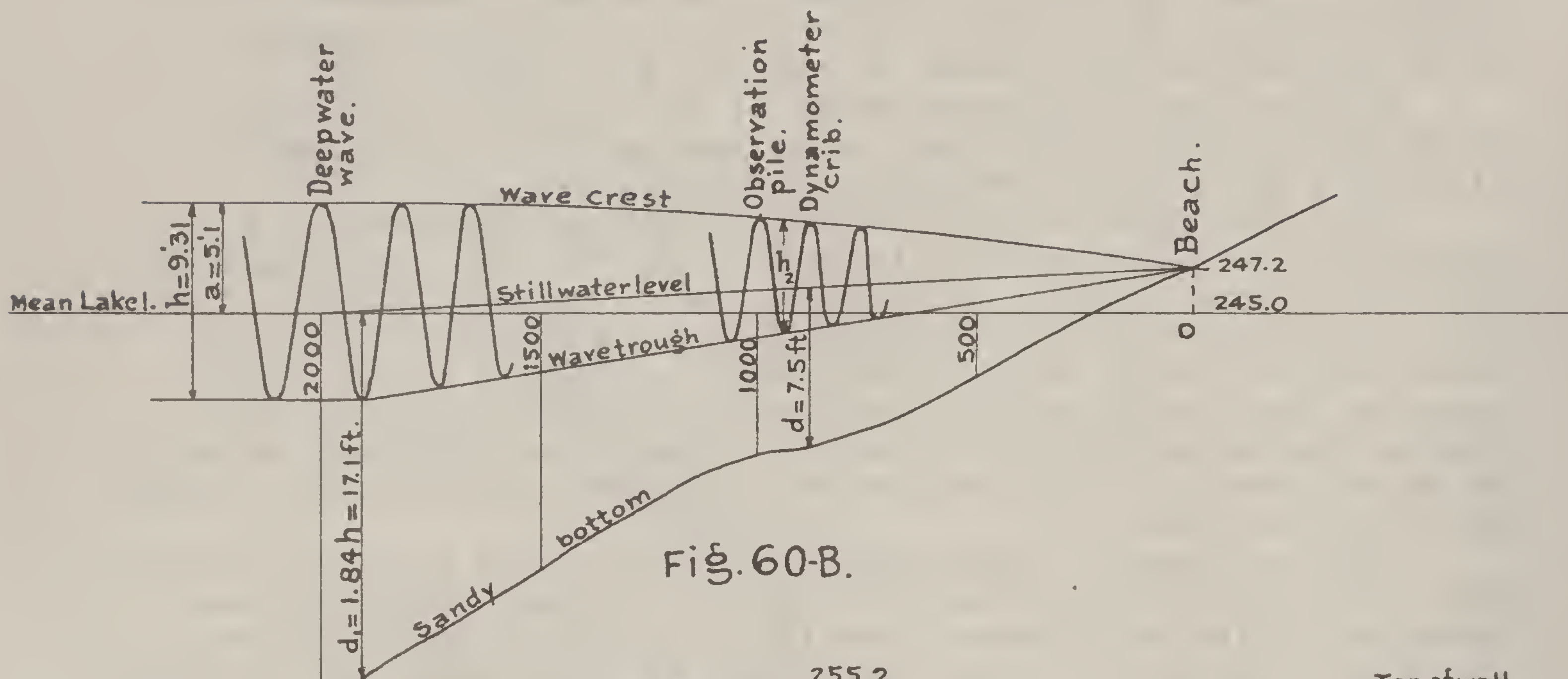


Fig. 60-B.

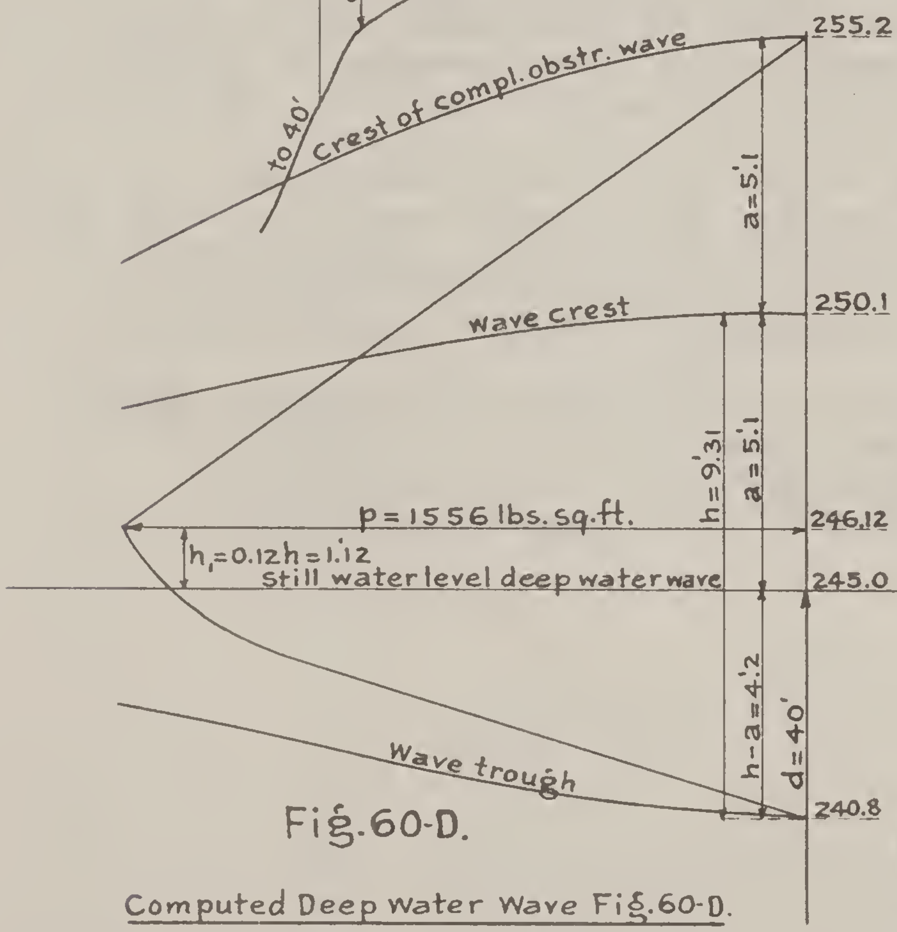


Fig. 60-D.

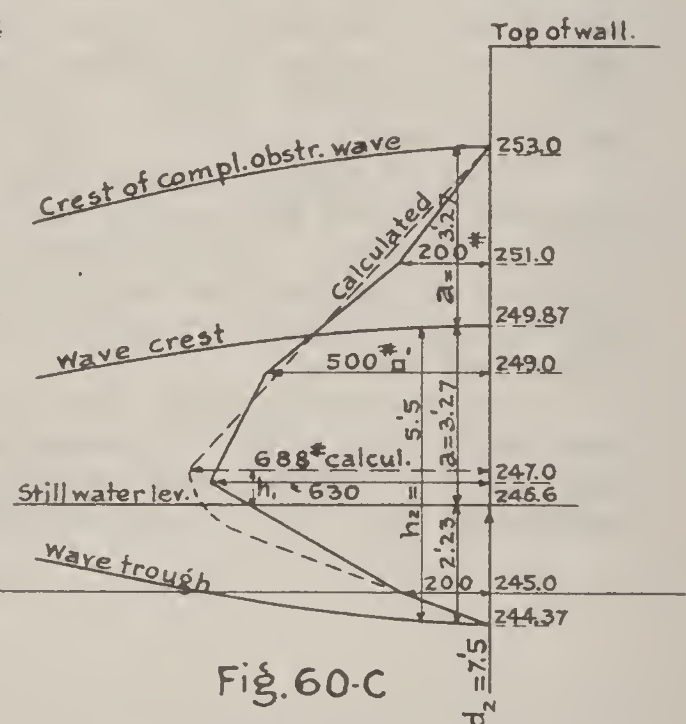


Fig. 60-C

Observed Wave Pressure.

Fetch 156 m. from E and 32 m. from S.E.
E. wind 40 m.p.hr., shifting to 30 m. S.E. when waves were at maximum.
$h_2 = 5.5$ ft., $L = 118$ ft., $v = 19.8$ sec. ft.

Computed Values.

$a = \frac{h_2}{2} + \frac{2h_2^2}{L} = 3.27$ ft., $h_2 - a = 2.23$ ft., $h_1 = 0.12h_2$.

$\frac{d_0}{L} = 0.068$, $C = 0.68$, $\mu = 1.66$ Tab. 59A.

making $v_2 = 2.26 \times 0.68 \sqrt{118} = 16.72$ sec. ft.

and $v_0 = 7.11 \times 1.66 \frac{5.5}{\sqrt{118}} = 6.28$ sec. ft.

also $v_2 + v_0 \ldots\ldots = 23.00$ sec. ft.

making $p = 1.3 \times \overline{23.0}^2 = 688$ lbs. sq. ft.

Eq. 59 D gives $v_2 = 0.9 v \sqrt[4]{\frac{7.5}{40}} = 17.45$ sec. ft.

with a reduced height $h_2 = 9.31 \sqrt{\frac{7.5}{40}} = 6.14$ ft.

Computed Deep Water Wave Fig. 60-D.

$V = 30$ m.p.hr., aver. $D = 100$ m., $h = 0.17\sqrt{VD} = 9.31$ ft.

$L = 21h = 195$ ft., $d_1 = 1.84h = 17.1$ ft., $a = \frac{h}{2} + \frac{h^2}{L} = 5.1$ ft.

$h - a = 4.2$ ft., $h_1 = 0.12h = 1.12$ ft.

For $d = \frac{L}{5} = 40'$ find $\frac{d_0}{L} = 0.22$, $C = 0.94$, $\mu = 1.06$

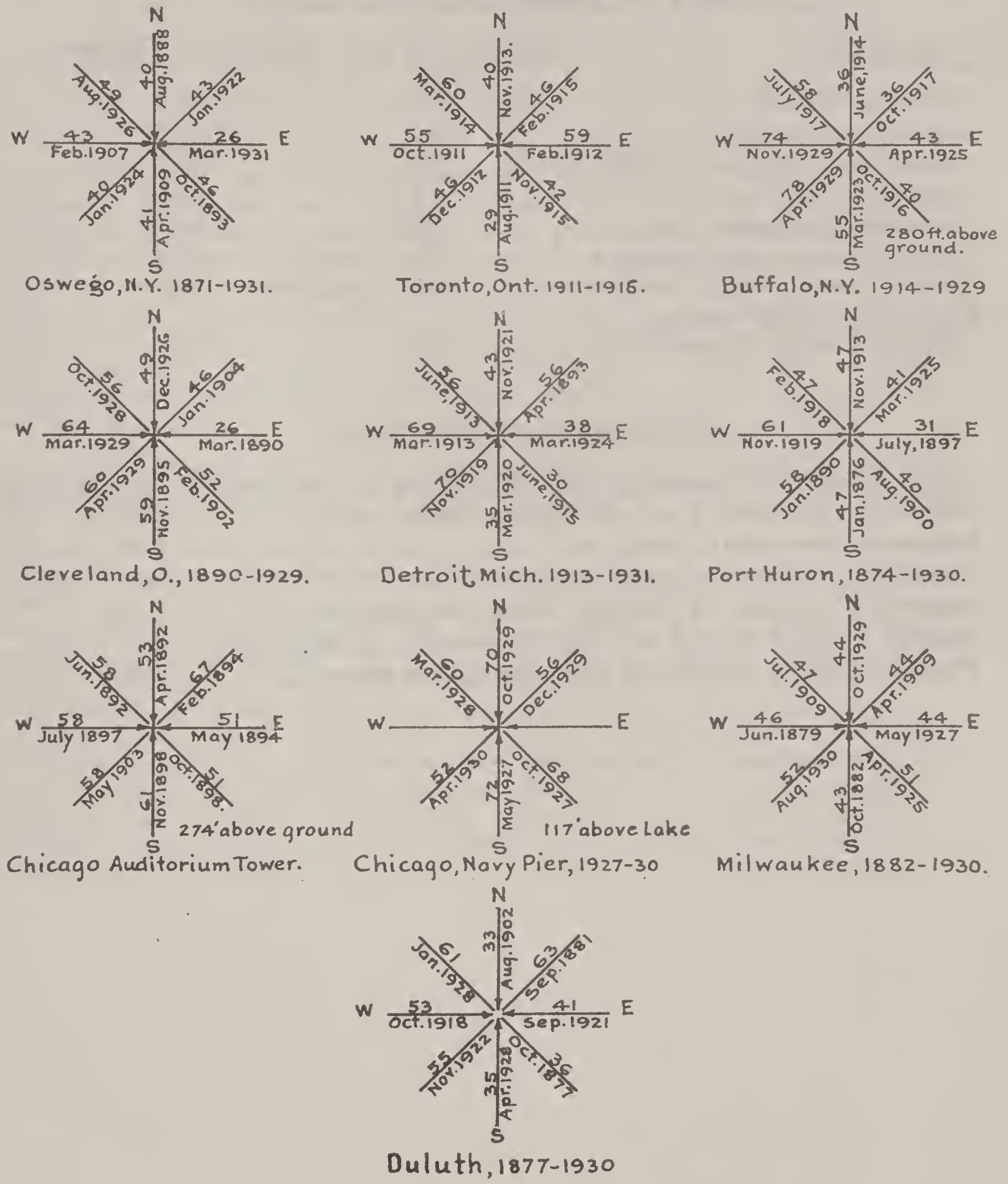
Then $v = 2.26\, C\sqrt{L} = 29.6$ sec. ft.

and $v_0 = 7.11 \mu \frac{h}{\sqrt{L}} = 5.0$ " "

also $v + v_0 = 34.6$ sec. ft.

making $p = 1.3(v + v_0)^2 = 1556$ lbs. sq. ft.

Wind Roses, showing maximum wind velocities for the 8 cardinal points, and the dates on which they occurred. Data obtained from U.S. Weather Bureau Officials in charge of the several stations.



All the above wind velocities are based on 3-cup anemometer values.

Figs. 60 E.

The following data regarding weights and friction will be useful in designing breakwater cribs.

Weights in pounds per cubic foot.

Material	Weight in Air	Weight in Water
Granite blocks	170	107.5
Limestone blocks	166	103.5
Limestone rip rap - 40% Voids	100	62.5
Concrete, plain	148	85.5
Concrete, reinforced	152	89.5
Timber, green as shipped	35	uplift - 27.5
Timber, wet, incl. bolts	50	uplift - 12.5
Average timber cribs, filled with limestone	92	50

Coefficients of friction

All surfaces of masonry or brickwork in contact	0.65 to 0.70
Stone or brickwork on moist unctuous clay	0.3
Stone on bed rock, dry	0.7
Concrete blocks on well wetted concrete floor	0.7
Concrete blocks on rubble base, submerged	0.65
Rubble filled cribs on rubble mound, submerged	0.9
Rubble filled cribs on bed rock, submerged	0.6

Art. 61 Stability of Breakwater Cribs subjected to Wave Force. The previous articles present the several steps in evolving the wave pressure curve from a given fetch and wind velocity, or in other words, of evaluating the wave effect produced by a certain cause. The ultimate aim is however, to design a breakwater crib of adequate dimensions to safely withstand the wave pressure to which it may be exposed during the most severe storms.

The complete solution of such problems will now be undertaken on the assumption that the exact location for a proposed structure and the maximum wind for the locality are definitely known. The examples chosen are the Breakwater at Harbor Beach, Mich., and Magann's Pier at Toronto, Ont., both of which structures have withstood many severe storms during the past 30 years without showing any marked structural weakness. Hence, if the method of analysis here given leads to findings in conformity with the actual behavior of these structures, then the speculative features involved in the method will have been reduced to a negligible quantity.

Problem 61-A. Stability investigation of Harbor Beach Breakwater. See Fig. 61-A and the computations which follow. The fetch, scaled from a map of Lake Huron, is found to be 132 miles north or 92 miles N.E. The latter fetch is about normal to the breakwater. The maximum wind velocity from the N. is taken as 60 miles per hour. Since waves usually swing more or less normally to the shore as the water becomes shallower, the larger wave is assumed to arrive normally in front of the wall. Eq.57-A then gives the probable height of wave as 15.13 ft. and Eq. 58-A gives its length as 212 ft. Since the water in front of the breakwater has a depth of 28 ft. this wave suffers no appreciable reduction due to shoaling effect. Hence the problem is comparatively simple and requires no further description aside from that given. In estimating the factor of safety no account was taken of the weight of water on top of the superstructure.

BREAKWATER AT HARBOR BEACH, MICH.

STABILITY INVESTIGATION.

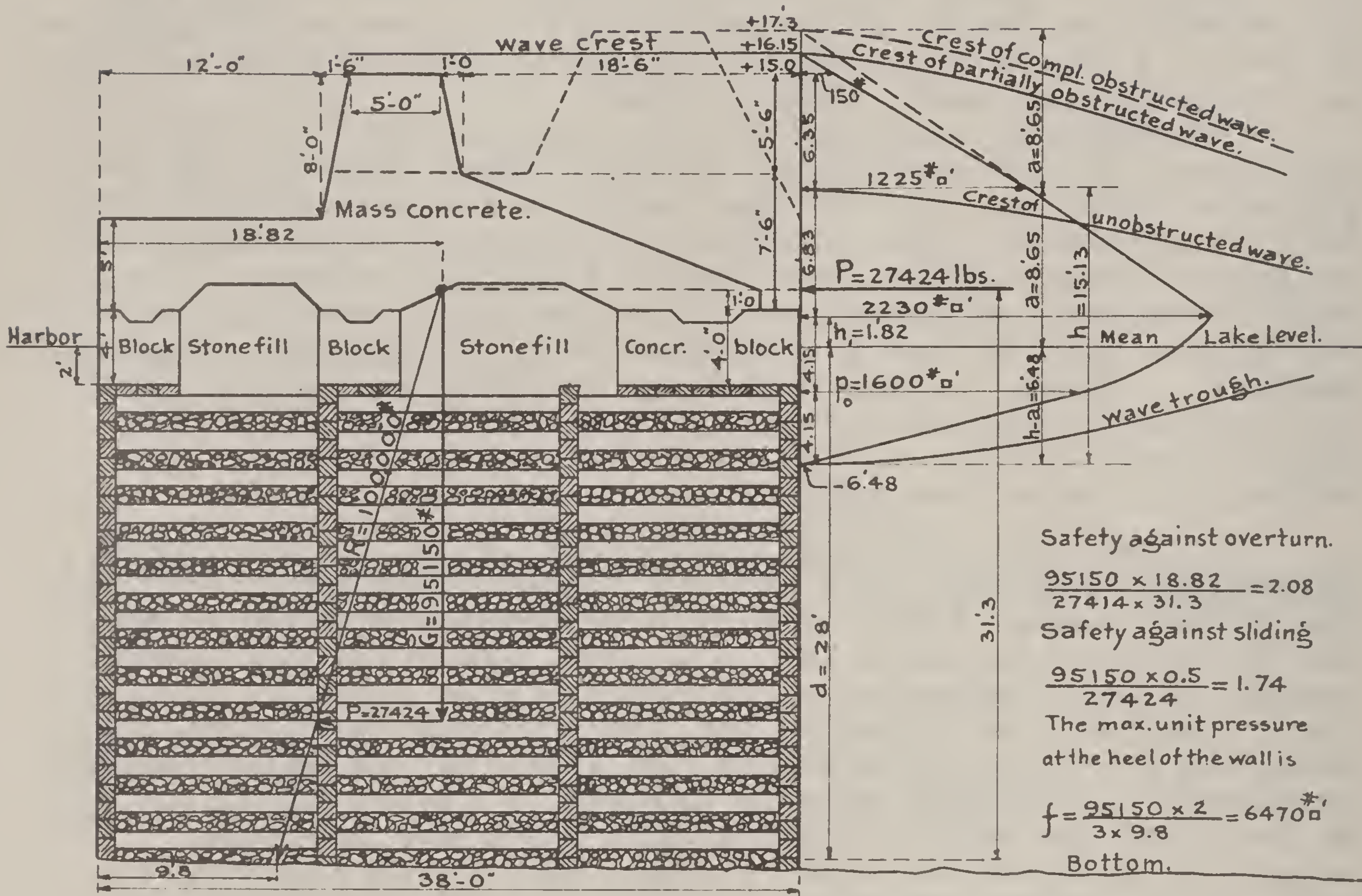


Fig. 61-A.

The fetch $D=132$ m. North; Angle $\alpha=90°$; wind $V=60$ m.p.hr. Hence the deep water wave by Eq. 57·A has $h=0.17\sqrt{VD}=15.13$ ft. and by Eq. 58A find $L=14.0h=212$ ft.; $d=28$ ft. The height of this wave above still water level by Eq. 58B is $a=\frac{h}{2}+\frac{h^2}{L}=8.65$ ft., and $h-a=6.48$ ft. This wave will break when the depth becomes $d_1=1.84h=27.84$ ft., hence the wave will strike the breakwater before breaking, and will suffer no material reduction due to shallow water.

Now find $\frac{d_0}{L}=\frac{d}{L}+2\left(\frac{h}{L}\right)^2=\frac{28}{212}+2\left(\frac{15.13}{212}\right)^2=0.142$, for which Table 59A gives $c=0.829$ and $\mu=1.21$. Then Eq. 59G gives $v+v_0=2.26c\sqrt{L}+7.11\mu\frac{h}{\sqrt{L}}=36.2$ sec. ft. and the maximum unit wave pressure becomes $p=1.7(v+v_0)^2=2230$ lbs. per sq. ft. acting at $h_1=0.12h=1.82$ ft. above still water level. Then make $p_0=0.72\,p=1600$ lbs. sq. ft.

This data determines the complete wave pressure curve as shown above, which requires slight modification for the portion of wave which is not obstructed. The total effective wave pressure is thus found as $P=27{,}424$ lbs. per lin. ft. acting 31.3 ft. above the bottom. The weight of the crib, allowing for buoyancy, is calculated as $G=95{,}150$ lbs. per lin. ft., which combined with P gives the resultant R acting on the base and producing a pressure of $f=\frac{95150\times 2}{3\times 9.8}=6470$ lbs. per sq. ft. at the heel. No account was taken of the weight of the water on top of the superstructure.

Referring again to Fig. 61-A, it will be noticed that the wave force P was combined directly with the weight G in testing the safety of the structure, without regard to the shape of the concrete deck or superstructure. This would be correct in testing the stability of the crib, provided that the entire wave impact is imparted simultaneously to all parts of the structure.

However, since the superstructure does not attain its full height at the face of the crib, but slopes gradually for a distance of 18 ft. and then presents another surface to resist the upper portion of the wave, it must be apparent that the full wave force is not expended simultaneously, nor with the pressure assumed in the analysis. The wave could not be built up to its full height until after it had advanced to the last obstruction a second after striking the first. In other words, the total wave energy is transferred to the crib during a greater period of time than was assumed in designing the wave pressure curve, and hence the shock is in reality less severe than the calculation shows.

This example illustrates how the wave pressure may be rendered less severe simply by adopting a superstructure design which does not obstruct the wave in its entirety on a vertical surface, by allowing the wave to build up gradually. However, this may develop some difficulties in the design of the concrete superstructure and no doubt will cause a larger proportion of the wave to jump over the wall. The back wash may also be retarded to such an extent as to meet the next oncoming wave at an unfavorable time. The wave force striking the high center wall will subject the latter to a moment of about 9600 ft. lbs. which might cause the mass concrete deck to break unless steel reinforcement is employed.

Considering the ample safety of the crib itself, it would appear that the high mass concrete wall might better have been placed over the lake front thus affording greater security for the concrete face blocks, besides adding very materially to the strength of the concrete deck where it receives the worst punishment.

In designing a breakwater superstructure, much depends on the purpose which it is to serve. It may be necessary to completely obstruct oncoming waves, especially if ships are intended to moore on the harbor side. In other cases it may not matter how much water is thrown over the wall so long as the wave force is rendered harmless.

The superstructure of the Harbor Beach breakwater, if intended to protect ships while tied on the harbor side of the breakwater, might have been designed as indicated by a dotted outline, Fig. 61-A, so as to obstruct more completely

the oncoming waves, at the same time increasing the resisting moment of the structure and adding strength to the deck.

The use of some reinforcing bars in connection with these superstructure designs is very desirable.

It is also necessary to provide breathing openings through the concrete deck to allow relief for air pressure which is formed in the body of the crib. Wave pressure will also set up hydrostatic pressure on the inside of cribs, thus exerting a bursting effect on all parts of the structure which must not be overlocked.

Problem 61-B. Stability Investigation of Magann's Pier, Fig. 61-B. This pier is located between Jameson St. and Dowling Ave., Toronto, Ontario, in 9 ft. of water, deepening to 11 ft. at 50 ft. out. The worst conditions exist for a S.E. wind with a fetch D = 30 miles, making an angle $\alpha = 68^{\circ}$ with the lake face of the pier. A wind velocity of 55 miles per hour was taken as a possible maximum.

With this data, the deep water wave dimensions may be estimated by Eqs. 57-A and 58-A, with a velocity of propagation v, figured by Eq. 59-C for a depth $d = L/5$. This depth is arbitrarily assumed because a deep water wave may continue into much shallower water before the shoaling effect becomes appreciable.

The important shoaling effect is produced when the wave advances from the depth $d = L/5 = 21$ ft., to that in front of the pier which is only 9 ft. The reduction in velocity and height, due to this shoaling, is evaluated by Eqs. 59-D and the pressure exerted by the wave so reduced is found from Eq. 59-G. Now since this wave does not strike the pier normally, a further pressure reduction is made as per Eq. 59-H, but this does not alter the wave height just found. The pressure area may now be drawn for a completely obstructed wave, reaching the height $2a = 5.62$ ft.

In this problem, the pier is not sufficiently high to completely obstruct the wave, thus necessitating a reduction in the pressure area for the portion of wave which is not obstructed. The final pressure P is represented by the area of the pressure curve between the wave trough and the top of the pier. The detail calculations are given following the drawing, and show a stability against overturning and sliding on the base quite in conformity with the behavior of this structure during its 30 years of existance.

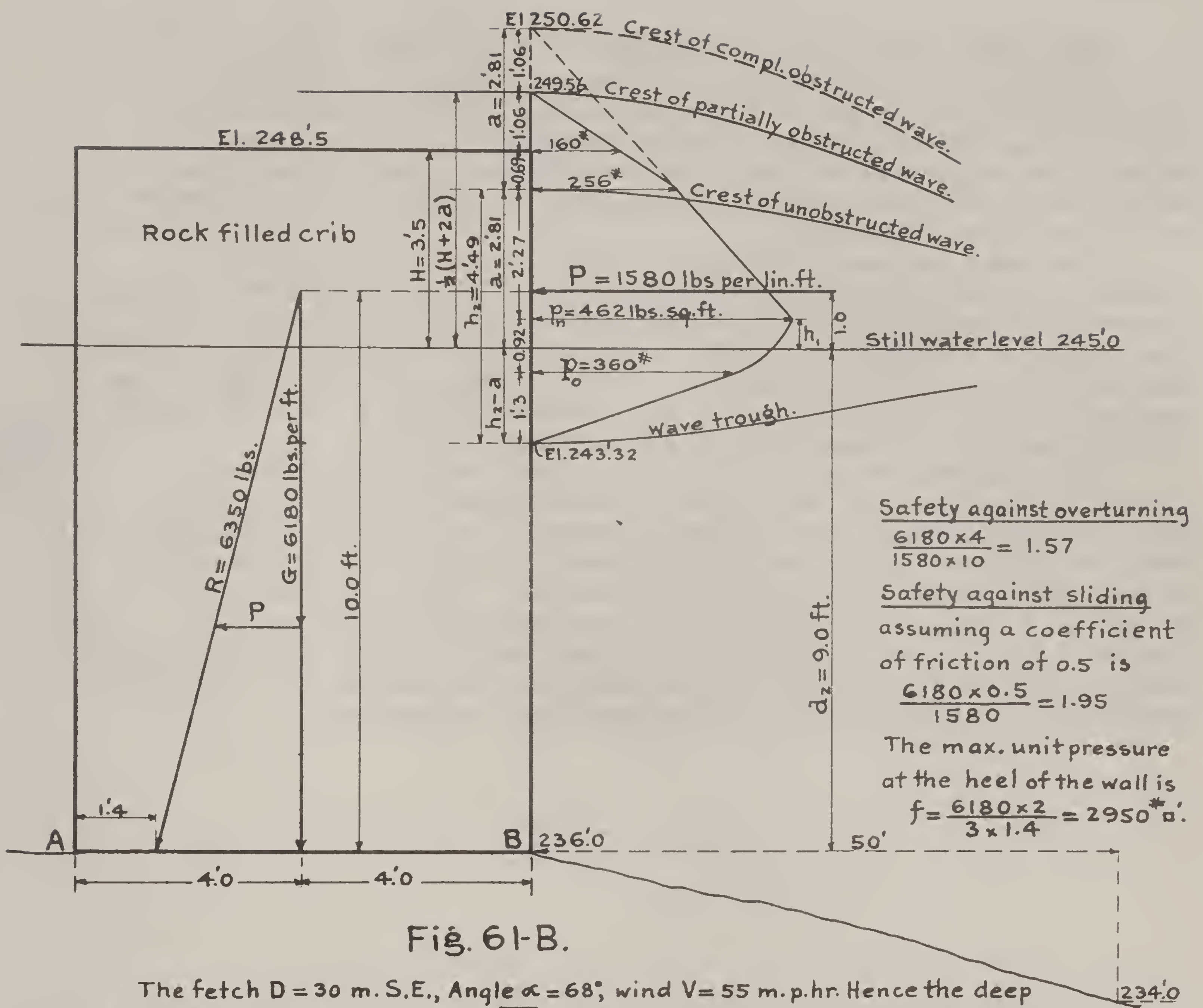


Fig. 61-B.

The fetch D = 30 m. S.E., Angle $\alpha = 68°$, wind V = 55 m.p.hr. Hence the deep water wave by Eq. 57-A, has $h = 0.17\sqrt{VD} = 6.90$ ft., and by Eq. 58A find $L = 15.3h = 105.6$ ft.

The depth in which a deep water wave may continue without appreciable effect on its height and length may be taken as $d = \frac{L}{5} = 21$ ft. Then for $\frac{d_o}{L} = \frac{d}{L} + 2\left(\frac{h}{L}\right)^2 = \frac{21}{106} + 2\left(\frac{6.9}{106}\right)^2 = 0.33$, find $c = 0.98$ in Eq. 59C, giving $v = 2.26\, c\sqrt{L} = 22.8$ sec. ft.

With these dimensions and velocity v as a deep water condition, we now encounter shallow water and the wave suffers a reduction in size and velocity. According to Eq. 58D, this wave will break when the depth is reduced to $d_1 = 1.84h = 12.7$ ft. Just before striking the pier, the depth is further reduced to $d_2 = 9.0$ ft. causing the initial velocity v to retard to a value v_2 according to Eq. 59D. Thus $v_2 = 0.9\,v\sqrt[4]{\frac{d_2}{d}} = 16.53$ sec.-ft., and the wave so retarded is reduced to a height $h_2 = h\sqrt{\frac{d_2}{d}} = 6.9\sqrt{\frac{9.0}{21}} = 4.49$ ft. with a length $L = 15.3h_2 = 68.7$ ft. The height of this wave above still water level by Eq. 58B is $a = \frac{h_2}{2} + 2\frac{h_2^2}{L} = \frac{4.49}{2} + 2\frac{4.49^2}{68.7} = 2.81$ ft., and $h_2 - a = 1.68$ ft. Also find $\frac{d_o}{L} = \frac{9}{68.7} + 2\left(\frac{4.49}{68.7}\right)^2 = 0.261$, for which Table 59A gives $\mu = 1.04$. Then by Eq. 59G find $v_2 + v_o = 16.53 + 7.11 \times 1.04 \times \frac{4.49}{\sqrt{68.7}} = 20.5$ sec. ft. The maximum unit pressure then becomes $p = 1.3\,(v_2 + v_o)^2 = 1.3 \times 20.5^2 = 546$ lbs. per sq. ft. acting at $h_1 = 0.12h_2 = 0.54$ ft. above still water level. This pressure resolved normally to the pier as per Eq. 59H becomes $p_n = p\sin^2\alpha = 546 \times 0.846 = 462$ lbs.

From the above data construct the final pressure curve as above indicated to obtain the total pressure P = 1580 lbs. per lin. ft. of pier. The weight of the crib, allowing for buoyancy, is G = 6180 lbs. per ft., which combined with P gives the resultant R acting on the base AB.

Art. 62. Concluding Remarks. In closing this subject the author wishes to emphasize that stability investigations of this character, or designs based on similar calculations, must be regarded as more or less speculative, yet the methods here presented for the first time, were developed after much thought and study and are considered as accurate as the subject warrants.

The same method of analysis was applied to existing breakwater structures at Presque Isle, Ontonagon, Grand Marais, Agate Bay, Portage Lake and Marquette, all on Lake Superior. The Buffalo breakwater comprising several designs with varied exposures was also analysed. In each case the results of the analysis agreed closely with the behavior exhibited by these structures during many years of service.

The western breakwater at Toronto, built in 1915-16, according to designs which were criticised at the time as being structurally inadequate, showed a factor of safety of about 4 against both overturning and sliding, according to an analysis made in 1915 by the author. This breakwater, covering a length of about 3-1/2 miles, has weathered all storms for the past 15 years without manifesting the slightest signs of weakness, again demonstrating the reliability of results obtainable with the method here described.

Art. Tank Bottom a Surface of Revolution. In this general case the bottom plate is curved in two directions with radii ϱ and n as shown in Fig.1. The radius ϱ is the radius of curvature of the plate at any point P for a vertical section. The radius $n=\frac{r}{\sin\alpha}$ is the normal to the surface at P, and is the radius of a meridian circle through P with its center on the axis of revolution at N. For a conical bottom, ϱ = infinity.

The principal stresses H and S, per unit length and width of plate at any point P, are related to the normal unit pressure p acting at P, and this fundamental relation will now be determined.

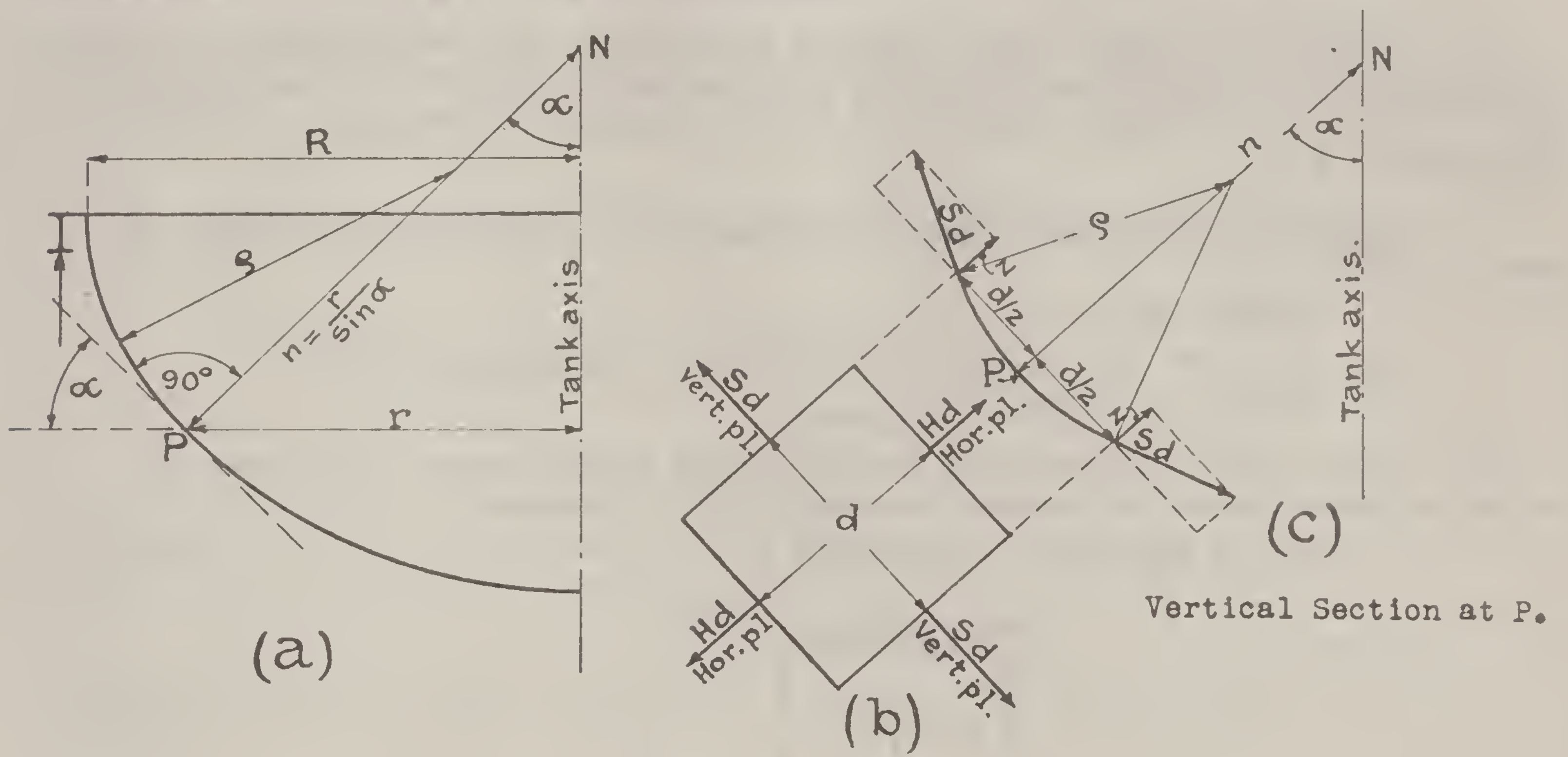


FIG.1.

For a unit pressure p on the plate, the total pressure on the small area d^2 will be pd^2. H and S are the stresses per unit edge dimension of plate in the two directions.

The tension along the edge of the plate area d^2 will be Sd in the vertical meridional plane, and Hd in the horizontal direction, as shown in Fig.1-b., where S and H are stresses per unit width in the two directions.

From Fig.1-c, by similar triangles, $\varrho : d/z = Sd : z$, giving $z = \frac{Sd^2}{2\varrho}$, which is the component of Sd in the direction of the normal pressure p at the center of the plate area d^2.

A similar value z' may be found for the forces Hd, acting horizontally, as $z' = \frac{Hd^2}{2n}$, where $n = \overline{PN} = \frac{r}{\sin\alpha}$.

Now since the sum $2(z+z')$, must equal the total pressure pd^2, then

$$\frac{Sd^2}{\varrho} + \frac{Hd^2}{n} = pd^2 \quad \text{or} \quad p = \frac{S}{\varrho} + \frac{H}{n} \quad \text{or} \quad H = n\left(p - \frac{S}{\varrho}\right) \text{ ---------------------- } (1)$$

which is a fundamental relationship between the unit pressure p at any point of the curved plate, and the stresses H and S, acting respectively in a horizontal and a vertical direction or plane.

Now it is always possible to evaluate the stress S per unit width of plate, in terms of the water load G, acting on the tank bottom and within the cylindrical volume of radius r. Then with S and p known for any point P, the horizontal stress at this point is given by Eq.1, as $H = n\left(p - \frac{S}{\varrho}\right)$.

Thus the vertical component of S is $S \sin\alpha$ per unit of length of plate, and acts around the entire circumference $2\pi r$ of the small circle with radius r. The total vertical shear $2\pi r S \sin\alpha$ around the circumference $2\pi r$, must then be equal to the vertical water load G contained within the cylinder of radius r.

Hence, $G = 2\pi r S \sin\alpha$ or $S = \dfrac{G}{2\pi r \sin\alpha}$ ---------------------- (2)

Equations 1 and 2 thus afford a solution for the two principal stresses at any point P in the bottom plate of a tank, shaped as a surface of revolution. When the bottom is conical, then ϱ is infinite and Eq. 1 gives $H = np$, which is independent of S.

Hemispherical Bottom Tanks. The volume of liquid of depth y, above the tank bottom is

$$\pi y^2 (R - y/3)$$

The volume of the cylinder of radius r and height h—y is

$$\pi r^2 (h-y) = \pi (2R-y)\, y(h-y).$$

Hence, the total weight G resting on the bottom within the circle of radius r, due to water weighing w lbs. per cu. ft., becomes:

$$G = w\pi\left[(2R-y)hy - Ry^2 + \tfrac{2}{3}y^3\right] \text{ -------------------------------- (3)}$$

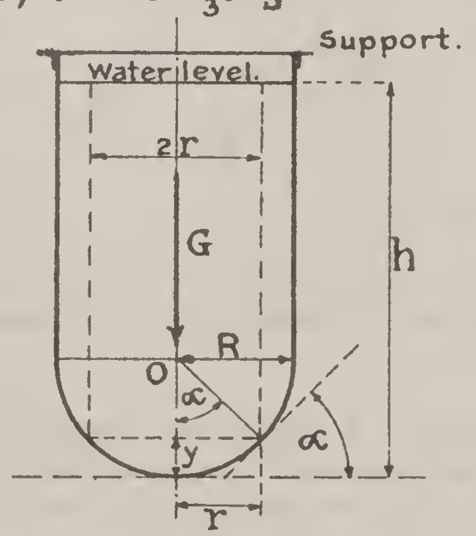


FIG. 2

Noting that $r \sin\alpha = \dfrac{r^2}{R} = (2R-y)\dfrac{y}{R}$, Eq(2) gives the value of S as:

$$S = \frac{\pi w\left[(2R-y)hy - Ry^2 + \frac{2}{3}y^3\right]}{2\pi(2R-y)\frac{y}{R}} = \frac{wR}{2}\left[h - \frac{y}{3}\left(\frac{3R-2y}{2R-y}\right)\right] \text{ ---------------------- (4)}$$

The value of S from Eq.4, substituted into Eq.1, and noting that $n = \varrho = R$ for the hemispherical bottom, then

$$H = pR - S = wR(h-y) - S = \frac{wR}{2}\left[h - \frac{y}{3}\left(\frac{9R-4y}{2R-y}\right)\right] \text{ ------------------------- (5)}$$

The stresses H and S become minimum when $y = R$, giving

$$S = \frac{wR}{2}\left(h - \frac{R}{3}\right) \quad \text{and} \quad H = \frac{wR}{2}\left(h - \frac{5R}{3}\right) \text{ ------------------------------------ } (6)$$

for all points at the level of the center O, or distant R above the bottom of the tank.

So long as the tank is full and $h - R/3$ does not become negative, then S remains a tensile stress. However, to avoid compressive values for H, the value of h must be equal to or greater than $5R/3$. This would require the tank volume, including the cylindrical portion, to be at least $\frac{4}{3}\pi R^3$ or

$$R \geqq 0.62\sqrt[3]{\text{Volume of tank.}} \text{ ---------------------------------- } (7)$$

When the water level recedes to a depth h, less than $5R/3$, then, according to Eq.6, H becomes negative or compressive, while just above the hemisphere $H = R(h - R)w$, which is tensile. This stress reversal at the top of the hemisphere should be safoguarded by a horizontal girder provided at this level.

When the water just fills the hemisphere with level at O, then $h = y = R$, and $S = \frac{wR^2}{3} = -H$, at the water level.

When the water level drops below O to a depth y, then

$$S = \frac{wRy}{6}\left(\frac{3R - y}{2R - y}\right) = -H \text{ -- } (8)$$

When there is a vertical stand pipe connected with the tank bottom, thus relieving the tank of its water load, then the above Eq.2 and all subsequent formulas will require modification to allow for that portion of the weight G which is then carried on a separate foundation under the stand pipe.

Art. Tank Supports. When a tank is hung from a circular wall, the matter of peripheral support is quite simple and resolves itself into providing a suitable bearing plate to transfer the uniform tank weight g to the masonry.

However, when a tank is supported on a tower, the weight must be carried by a balcony girder to the tower legs and thence to footings in the ground. These structural elements will also be called upon to carry the lateral wind pressure acting on the tank and the tower legs.

The balcony girder is considered fixed at the column supports A, has a radius R, and subtends a center angle 2α, Fig.3.

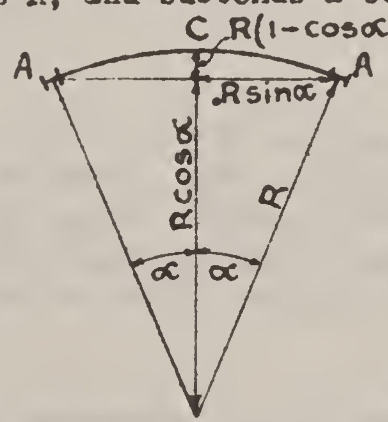


FIG.3.

Let G = the total weight of tank and contents carried on n tower legs.

$g = \frac{G}{2\pi R}$ = load per lin. ft. of girder.

$P = 2g\alpha R = \frac{G}{n}$ = the load carried by one column.

Then the bending moment M_c at the center of the beam, considering both ends fixed, is approximately

$$M_c = \frac{g}{24}(2R\sin\alpha)^2 = \frac{\sin^2\alpha}{6} gR^2 \text{ ----------------- } (9)$$

The bending moment M_a at the fixed end A is approximately

$$M_a = M_c\cos\alpha - gR^2(1 - \cos\alpha) \text{ ----------------- } (10)$$

Also the torsional moment T_a at the fixed end A is approximately

$$T_a = gR^2(\alpha - \sin\alpha) - M_c\sin\alpha \text{ -- } (11)$$

For tank towers, there are relatively few cases to be considered and the following Tables will avoid the necessity of evaluating the above formulas.

Table α Functions for Balcony Beams.

No. Posts in tower	α Degrees	α Arc	$\sin\alpha$	$\cos\alpha$	$1-\cos\alpha$	$\alpha-\sin\alpha$
4	45	0.7854	0.7070	0.7070	0.2930	0.0784
6	30	0.52[illegible]	0.[illegible]0	0.8660	0.1340	0.0236
8	22.5	0.3928	0.3827	0.9239	0.0760	0.0101
12	15	0.2619	0.2588	0.9659	0. 0340	0.003[illegible]

Table Bending Moments M_c and M_a and Torsional Moments T_a.

NO. Posts in tower	Center mom. $M_c = \frac{1}{6}\sin^2\alpha\ gR^2$	End moment $M_a = M_c\cos\alpha - g\ R^2(1-\cos\alpha)$	Torsional mom. $T_a = \pm g\ R^2(\alpha - \sin\alpha) - M_c \sin\alpha$
4	0.0833 gR^2	0.2340 gR^2	0.0195 gR^2
6	0.0417 gR^2	0.0980 gR^2	0.0027 gR^2
8	0.0244 gR^2	0.0536 gR^2	0.0008 gR^2
12	0.0112 gR^2	0.0[illegible]3 gR^2	0.0002 gR^2

The required section modulus S for the balcony beam is given by the approximate formula:

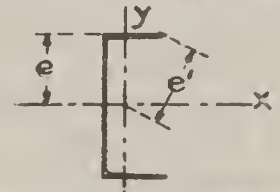
$$S = \frac{M_a}{f}\left[1 + 2.9\left(\frac{T_a}{M_a}\right)^2\right] \text{for } M_a \text{ and } T_a \text{ in inch pounds,} \text{------------} (12)$$

where f is the allowable unit stress in pounds per square inch. The formula gives good results for $T_a < \frac{M}{8}$.

Also, the combined stress f at A, due to bending and torsion, is given by the approximate formula

$$f = \frac{M_a e}{I_x} + \frac{T_a e'}{I_x + I_y} \quad \text{for } M_a \text{ and } T_a \text{ in inch pounds,} \text{------------} (13)$$

where I_x and I_y are the principal moments of inertia of the beam, while e and e' are distances from the c. g. to the extreme edges of the section for bending and torsion respectively.

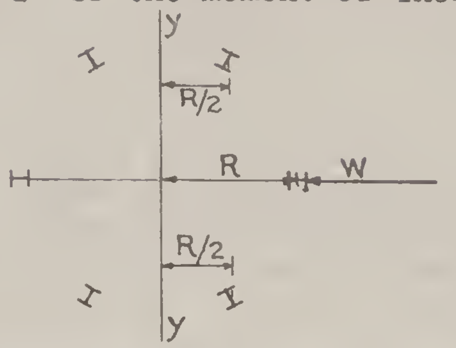


The connection between two adjacent balcony beams to the top of a column, will transmit a bending moment to the column equal to 2T , and due to elastic deformations in the several balcony beams, will tend to bulge the cylindrical tank plates out between column supports. Hence, the necessity for a heavy stiffening element at the top of the cylinder to maintain the circular shape.

The column load V will consist of the dead load $G/n = P$, combined with the transferred wind load. The vertical component of the transferred wind load is downward on the leeward columns and upward on the windward columns, and is maximum when the direction of the wind is taken parallel to a long diameter of the tower. Thus, for a wind moment M, Fig. 4,

$$V = P \pm \frac{M\,R}{I}$$

where I is the moment of inertia of the number of columns. Thus:



for 4 cols.	$I = 2R^2$	and	$V = P \pm \frac{M}{2R}$
6 "	$3R^2$		$P \pm \frac{M}{3R}$
8 "	$4R^2$		$P \pm \frac{M}{4R}$
12 "	$6R^2$		$P \pm \frac{M}{6R}$

Plan of column Footings.
FIG. 4 .

The wind moment $M = W h$, where W is the total wind force acting on the vertical projection of the tank and tower, and h is the height of the center of gravity of this projected area above the footing level on the ground.

The stress in the column ~~makes~~ will be $V/\cos\alpha$, where α is the inclination which the column makes with the vertical.

STRESSES IN LONG SPAN 3-HINGED ARCH ROOF TRUSSES +

By David A. Molitor, C. E.

Introductory. This subject is usually dismissed in our text books with a few remarks on wind forces and reaction conditions peculiar to the 3-hinged arch, followed by a statement that the stresses in the members may be found by any of the common methods, preferably by drawing a Maxwell stress diagram.

For long span, shallow, trusses with many short web members, the Maxwell diagram is practically useless, and the method of moments involves many long lever arms with moment centers outside the limits of the drawing, so that lengthy computations become necessary.

The method presented herewith was evolved by the writer in an effort to obviate these difficulties thereby affording a simple process of stress analysis with some important checks. It consists (1) of finding the reactions and resultants of the external forces by well-known methods; (2)-then computing the chord stresses from the resultant polygon by the method of moments; and (3) deriving the web stresses from the previously computed chord stresses by a new graphic method.

The reactions and resultants are best found as follows: For diagonal wind loads, find the resultant of the external forces by means of a force and trial equilibrium polygon and then draw the final resultant polygon which gives the actual reactions and the successive resultants of the external forces from the crown hinge to the abutment hinge. The chord stresses may then be computed. For vertical Loads, compute the resultant of the external forces for the half arch, also the horizontal thrust and the vertical component of the end reaction, after which the final resultant polygon may be drawn directly without any trial equilibrium polygon.

The following is a detailed description of the several steps comprising a complete stress analysis of the left half of a span for diagonal wind loads as illustrated on the accompanying diagram.

The Reactions. The wind loads w_1 to w_6 are combined into a force polygon, and a trial equilibrium polygon AC' is drawn with an assumed pole O', thus locating the resultant W, and the straight closing line AC'. A line through the pole O', parallel to AC', will divide the resultant A"C"= W into two reaction forces P_a and P_c. The resultant W must pass through the point A', which is the point of intersection of the two end rays of the trial polygon. The resultant W is thus found in direction and magnitude.

A final resultant polygon may now be drawn through the points A and C. To make this possible, the true pole O must first be located in the force polygon.

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The direction of the pole p must be parallel to the closing line AC, and the reaction R_2 acting at C, must pass through the right hand hinge of the arch. Hence, draw C"O through C" and parallel to R_2; also draw p from e, parallel to AC, giving the intersection O as the true pole.

Now draw the resultant polygon from A to C, with the true pole O, giving the actual end reactions R_1 and R_2, which must intersect in a new point on the resultant W as a check.

The position of any resultant of externally applied loads to the left of C is thus located for any section of the arch, and the magnitude of any such resultant may be scaled from the force polygon.

The chord Stresses can now be computed by the method of moments. Thus for member FG, the center of moments is n and the resultant in the bay F is 20.7 kips, acting with a lever arm 22.1 feet. Also, the lever arm from n to the member is 14.5 feet. Hence the stress FG = $20.7 \times \frac{22.1}{14.5} = 31.6$ kips. Similarly all other chord stresses may be computed.

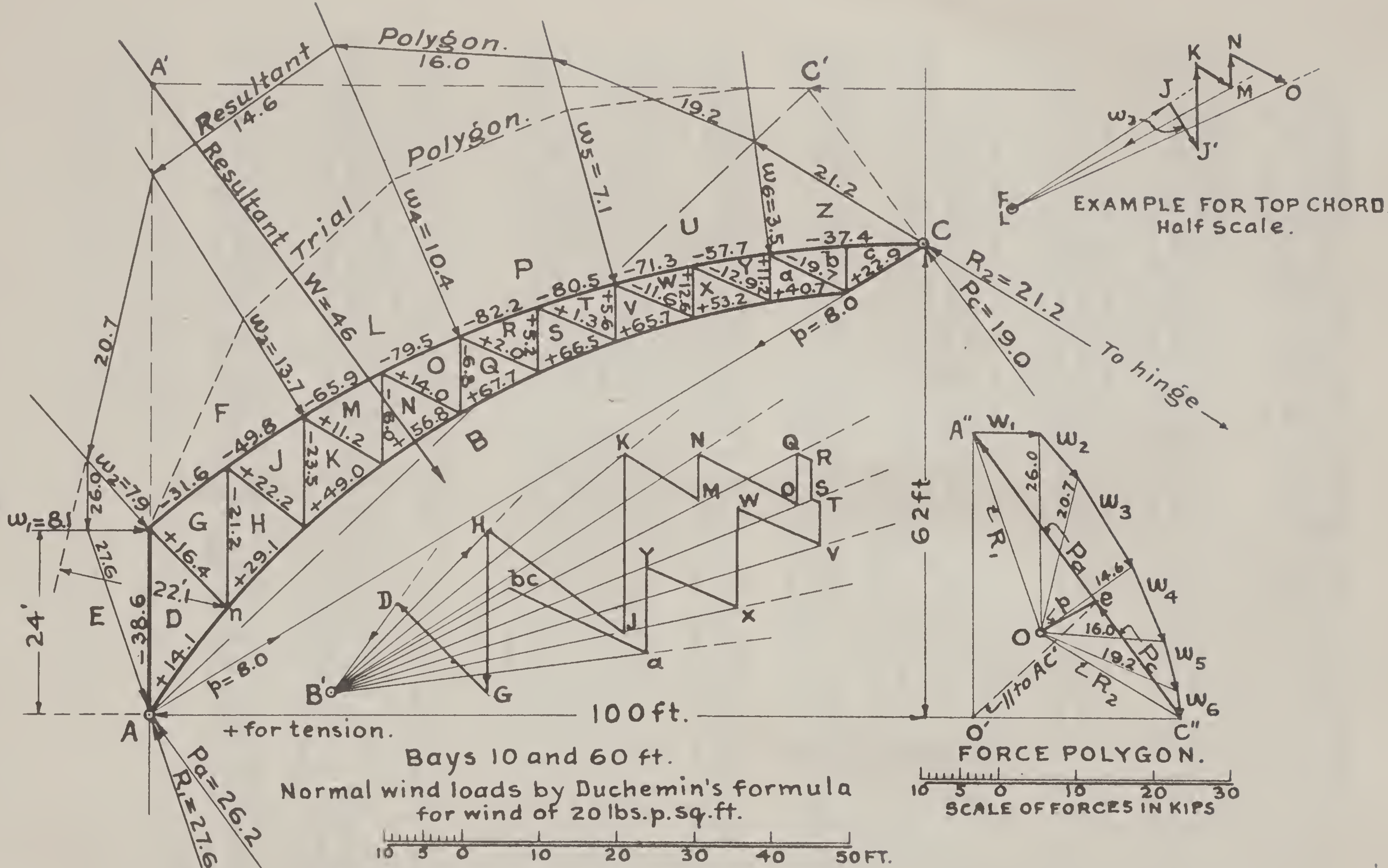
The Web Stresses are now derived from these chord stresses, obtaining one set of values from the bottom chord and a duplicate, or check, set of values from the top chord.

In the example here given, all the web stresses are found from the bottom chord stresses, while only a few are derived from the top chord to show how to insert the panel load w_3.

This method of determining the web stresses has the advantage of simplicity and accuracy, and affords a check not only for the web stresses but also for the previously computed chord stresses, which latter require no checking until a disparity in any one pair of results for a certain web member is discovered. Any erroneous chord stress is thus located. The method in its entirety is practically free from cumulative errors which are predominent in the ordinary stress diagram.

To derive the web stresses from the computed bottom chord stresses, proceed as follows: From any point B' as a pole, draw a series of rays parallel to the bottom chord members, successively from A to C. When great accuracy is desired, this can be done by coordinate plotting, since for any large problem the coordinates of the panel points should always be computed before proceeding to a final stress analysis.

The computed bottom chord stresses are now laid off on the respective rays B'D, B'K, B'N, etc., using any convenient scale of forces, in this case the same scale as used in the force polygon.



WEB STRESSES DERIVED FROM BOTTOM CHORD.
All stresses and loads in kips of 1000 lbs.

D. Molitor.
Feb. 1926.

The web stresses must now fit in between the points D, H, K, N, etc., so that all the members around a given panel point as n, will appear in a counter-clockwise order, thus DB', B'H, HG, and GD, to form a closed polygon. In each such partial stress diagram all stresses are known except the two web stresses, which are thus determined in direction and amount.

The direction of each stress is found by following around the partial stress diagram, starting with the two chord members of known direction. Sometimes this direction may be clockwise, as when the bottom chord members are in compression. In the case where some of the chord members of the same chord are in tension while others are in compression, it is necessary to draw the tension chords in a positive direction from the pole B' and the compression chord rays in the opposite direction.

The stresses in the web members JK, KM, MN, and NO have also been derived from the loaded top chord, merely as an illustration. The order in which the members ML, FJ and the load w_3 all arranged is immaterial so long as the known quantities are grouped together, permitting the two unknowns to close the polygon. The example shows a grouping which retains the system of rays from a common point as was done for the unloaded bottom chord.

When dealing with vertical loads only, the horizontal thrust is computed and the final resultant polygon is drawn without the trial polygon. To compute the horizontal thrust, take moments about C of all forces to the left of C, including the vertical reaction at A. The sum of these moments divided by the rise of the arch, gives the horizontal thrust.